THE
MATHEMATICAL
GAZETTE

EDITED BY
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WITH THE CO-OPERATION OF
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LONDON
G. BELL & SONS, LTD., PORTUGAL STREET, KINGSWAY, W.C. 2.
AND BOMBAY

Vol. XII., No. 176. MAY, 1925. 2s. 6d. Net.

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The Mathematical Association.

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THE LORENTZ TRANSFORMATION.

BY W. C. FLETCHER, M.A.

THE methods usually adopted for demonstrating the Lorentz Transformation are algebraic. Algebra is a powerful means of arriving at results, but does not in itself explain those results. In the present matter, the average reader is left with a sense of puzzledom, baffled by the paradoxes evolved by the algebra; the results are demonstrated, but he is unsatisfied. The following paper is an attempt to treat the matter from first principles, avoiding algebra and getting down to the plain common sense. The results are obtained quite simply; paradox is avoided. Some observations are hazarded at the end on the questions of simultaneity, "before" and "after," in which the view taken differs somewhat radically from that ordinarily adopted.

Objection is commonly taken to any use of the aether as a system of reference on the grounds (1) that the Michelson-Morley and other experiments have shown that we cannot detect it, and (2) that it is unnecessary, as the Lorentz Transformation can be established without it.

The purpose of the present article, however, is to dispense with the algebraical processes by which the Transformation is commonly established and to expose in the simplest fashion the meaning of the terms that occur therein. In order to do this it is necessary to follow the track of light signals, and in the first instance at least it is convenient to make use of the aether for the purpose. Later we will consider how far the aether can be dispensed with.

For the difficulty is this: suppose that two material worlds S_1, S_2 are moving apart with uniform velocity and that O_1B_1, O_2B_2 are two pairs of stations in those worlds which momentarily coincide at OB, and then separate. At the moment of coincidence let a flash be emitted at O, which after reflection at B_1 or B_2 returns to O_1 and O_2 respectively. How has the light travelled? There must be two separate paths, for the initial points coincide and the terminal points do not. If either world claims that it is at rest and that its share of the light travelled the same straight line forwards and backwards, the other will contradict and make the same claim for itself; while a third would contradict both.

To clear the issue and consider without prejudice what really happens, and so penetrate to the meaning of the transformation, we must assume that the light

travels in neither world, *i.e.* that both worlds are or at least may be in motion relative to the light. For this purpose and for this purpose only we will make use of the aether—as a medium or locus for the transmission of light which is independent of the material world in which the light manifests itself at certain points. The velocity of light in this aether is regarded as uniform and is denoted by c.

Consider then a single world S, moving through this aether with uniform velocity u, and suppose that it sets up a system of time and length measurements as follows. At a central station O, light flashes can be emitted and received back after reflection at other stations A, B. A single reflecting station B is established first (at some great distance, say 100,000 miles from O) and OB is taken as the fundamental unit of length. Also the time between emission of a flash at O and its return after reflection at B is taken as double the unit of time—which we will call a second. This being done, other stations A ... are so placed that a single flash at O returns to O from them all simultaneously. Then it will be assumed in the world that $OB=OA=\ldots$. The system can be extended indefinitely. Emission apparatus will be placed at $B, A\ldots$, and using O as the reflecting station, observers at $B, A\ldots$ will determine the "second" for themselves, and will then fix other stations more remote from O till the whole world is covered.

This light apparatus then affords the fundamental element (the regulator) of a clock—we will speak of it as the "light clock." For convenience mechanical clocks will be constructed, but they will be rated by the light clocks and will not be trusted without constant reference to these.

In assuming that $OB=OA=\ldots$ the observers assume that light travels *in their world* with uniform velocity in all directions, and we will further suppose that as the world has no knowledge of its motion through the aether, that it is assumed that that velocity is c.

The light clocks all have the same rate, but it is necessary to synchronise them. For this purpose O emits a flash one second before zero, and the observers at the first ring of stations $BA\ldots$ set their clocks to zero when they receive it—and the process is extended as before. The whole world is now supplied with clocks uniformly rated and as is supposed properly synchronised; but let us see what has really happened.

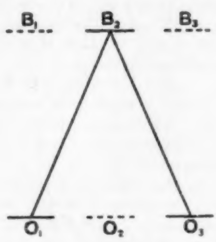

FIG. 1.

Take first the pair of stations OB, lying across the line of motion. The light did not travel from O to B and retrace its path, but started at O_1, was reflected at B_2 and returned to O_3. If we call $OB=L$ the half-time of transit is given by

$$c^2T^2=u^2T^2+L^2\,;$$

$$\therefore\ T=\beta L/c, \quad \text{where} \quad \beta^2=1/(1-u^2/c^2).$$

That is, the "second" is β times as long as the observers suppose.

Next consider a downstream station A. The light now follows the track

FIG. 2.

$O_1A_2O_3$, and if $OA=l$, and the times in O_1A_2, A_2O_3 are t_1, t_2, we have

$$ct_1=l+ut_1, \quad ct_2=l-ut_2;$$

$\therefore$ if $2t$ is the whole line of transit $t=\beta^2 l/c$.

But A and B were so placed that $t=T$;

$$\therefore \ l \text{ is not equal to } L, \text{ but } l=L/\beta.$$

That is, A and B are not (as the observers supposed) equidistant from O, but $OA=OB/\beta$.

This is the Fitzgerald contraction; owing to the motion through the aether, the world and all that is in it (including of course measuring rods), contract along the line of motion to $1/\beta$ of their length when lying across that line.

Our second result then is that the downstream unit of length is only $1/\beta$ of the cross-stream unit.

Last, consider the synchronisations. As to O and B, there is no difficulty; the times in O_1B_2 and in B_2O_3 are equal and the synchronisation is correct.

Not so that between O and A. O emits its synchronisation signal L/c before zero, for he calls $OA=OB=L$, though the true length is $OA=l=L/\beta$, but the time it takes to reach A_2 is $t_1=l/(c-u)=L/[\beta(c-u)]$, and in S units this is $L/[\beta^2(c-u)]=L(c+u)/c^2$;

$$\therefore \ A \text{ sets his clock } \textit{late} \text{ by } Lu/c^2.$$

That is, if A is a distance x_1 downstream in S units, the clock there is late by ux_1/c^2 seconds in its own units. Similarly, the upstream clocks are fast.

It is now easy to establish the transformations. S says that an event happened at the point $x_1y_1z_1$ at the local time t_1. How is this statement to be regarded from the point of view of an imaginary observer in the aether? He will say first of all, "Your local clock was slow by ux_1/c^2 of your own seconds, so your central time was t_1+ux_1/c^2; your second is β of mine; so, accepting your zero, I write $t=\beta(t_1+ux_1/c)$. Your downstream unit is only $1/\beta$ of mine, so I write x_1/β for your x_1, and you have moved bodily a distance ut away from me, so I write $x=x_1/\beta+ut$, and from the two equations I deduce $x=\beta(x_1+ut_1)$. Your y_1 and z_1 I accept, and write $y=y_1$, $z=z_1$."

We shall return to the last point, that of the cross-stream measurements, later.

We have then the usual equations:

$$t=\beta(t_1+ux_1/c^2),$$
$$x=\beta(x_1+ut_1),$$

and solving, we get

$$t_1=\beta(t-ux/c^2),$$
$$x_1=\beta(x-ut),$$

and we have quite simple explanations of the details: the curious term ux_1/c^2, due to the faulty synchronisation, the lengthening of the "second," the shortening of the downstream unit.

Let us now discard the aether for the time being and consider two material worlds S_1, S_2 having a relative velocity u. The whole of the previous argument applies with certain modifications. Each world regards itself as at rest and the other as in motion; we are free to take either as our standpoint, S_1

suppose. Then from that standpoint we treat S_1 as the aether, *i.e.* as the locus in which light moves; we regard the S_1 observations as normal or correct, and those of S_2 as vitiated by its motion; the results are as before: the S_2 second is β times the S_1 second; the S_2 downstream unit is $1/\beta$ of the cross-stream unit, which is also the general unit of S_1; the S_2 clocks are late or fast according to their position, and we have

$$x_1=\beta\{x_2+ut_2\}, \quad t_1=\beta\{t_2+ux_2/c^2\}.$$

Also the reversed equations

$$x_2=\beta\{x_1-ut_1\}, \quad t_2=\beta\{t_1-ux_1/c^2\}.$$

We arrived at these equations by adopting S_1 as the rest-system, and u is the velocity of S_2 as measured by S_1.

The reversed equations, however, suggest that $-u$ is also the velocity of S_1 as measured by S_2. This is very remarkable, for though the velocity is certainly one and the same (except as to sign) the units in which it is measured by the two are different and we should have expected different measures. How does it happen?

The answer is that the difference of velocity units is here compensated by the discrepancies in the clocks. Let us take the standpoint of S_2; he, of course, applies to S_1 all the criticisms that S_2 makes upon him and he says: "You say that my velocity is u; first of all your second is β of mine, then your downstream unit is $1/\beta$ of mine, so I correct your u to u/β^2. But when you say I have a velocity u, you mean that I travelled from your O to a point A, a distance u downstream (which I call upstream, for it is really you who are moving away from me) in 1 second; but your clock at A is fast by $u/c^2 \cdot u=u^2/c^2$, so what you called a second was only $1-u^2/c^2$ of your own seconds; so I finally correct your u to $u/[\beta^2(1-u^2/c^2)]$, which is, in fact u; so that, though I reject your units and your clock synchronisation, in this particular case of our mutual velocity, I accept your measure."

The more general interpretation of velocities as measured by one world in terms of the other follows just the same lines, but it is so instructive that it is perhaps worth repeating the argument. Suppose then that S_2 attributes to points in his world velocities v_1, v_2 along and across the stream; what will S_1 make of the statements?

As to v_2 he has little to say, merely that the second is too long, therefore his value of v_2 is v_2/β.

But as to v_1 he says: "Your second is long and your foot is short, so I correct to v_1/β^2; but your statement means that a point travels from a point C to a point D, a distance v_1 further downstream, in one second; but the clock at D was slow on the clock at C by $u/c^2 \cdot v_1$; therefore for your second I take $1+uv_1/c^2$ and write the velocity $v_1/[\beta^2(1+uv_1/c^2)]$. To this I add your general velocity u, and get for the velocity relative to myself $(u+v_1)/(1+uv_1/c^2)$.

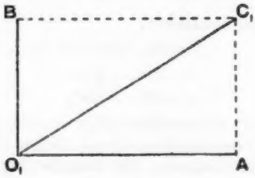

FIG. 3.

That it would not be right merely to add v_1 to u, is no more strange than that we cannot add, say, 5 metres to 4 yards, without reducing both to the same unit.

If a single point has velocities v_1, v_2, *i.e.* if these are the components of an oblique velocity, v_2 is no longer transformed by the mere substitution for it of v_2/β. For now the point travels, say, from O_1 to C_1 in one second by the clocks, and the clock at C_1 is slow on that at O just as is the clock at A, viz. by uv_1/c^2.

$\therefore$ *both* components, so to say, get the advantage of the lengthened second $1+uv_1/c^2$, and the corrected value of v_2 is $v_2/[\beta(1+uv_1/c^2)]$.

But now S_2 has a rejoinder to make: "You translate my downstream velocity v_1 into $v_1(1-u^2/c^2)/(1+uv_1/c^2)$; then you will translate an equal and opposite velocity $-v_1$ into $v_1(1-u^2/c^2)/(1-uv_1/c^2)$; so that you deny my power of recognising the equality, that is too absurd!" "Yes," says S_1, "that is so; how can we settle the matter? Suppose you endow two equal masses with your equal and opposite velocities, and see whether, on collision and coalescence, they bring one another to rest." S_2 does so and triumphantly succeeds. "Yes," again says S_1, "but are you sure that your masses were equal? I don't think they can have been. How do you test their equality?"

Then it is clear that they are working in a circle, that there is no test of equality of masses, independent of equality of velocities; and they must agree to differ on both points; velocities which S_2 regards as equal and opposite S_1 will regard as being in the ratio $1/(1+uv_1/c^2) : 1/(1-uv_1/c^2)$, and masses which S_2 regards as equal, S_1 will regard as being in the ratio $1+uv_1/c^2 : 1-uv_1/c^2$.

This, of course, is not a complete account of the treatment of mass, but it does show, simply and clearly, why the curious factor $1+uv_1/c^2$ comes into the transformation of mass from one system to another; namely, because the reciprocal factor comes into the transformation of velocities, and that again because of the differences in synchronisation.

Reverting now to our original consideration of a single world moving through the aether, there was a point left over for discussion. Why did the aether observer accept without criticism the world's cross-stream measurements and write $y=y_1$, $z=z_1$?

Merely to explain the result of the Michelson-Morley experiment it would have served equally well to assume, *e.g.* a transverse expansion instead of a longitudinal contraction. If we are dealing with a single world the question has no meaning, for "absolute" size probably means nothing to us. But if we consider two worlds the question becomes definite and intelligible.

Suppose, then, an observer in S_1 observes and measures an object PQ in the world S_2. The method he must use is this: the central observer say, at O_1, observes the time of transit of P at his own station—call this zero. Other observers note the time of transit of Q, and one of them, say at C_1, reports this as zero; then the length of PQ will be taken as O_1C_1. Let us examine the

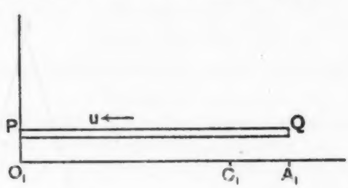

FIG. 4.

consequences. If PQ lies across stream there is no difficulty, for C_1 then lies on the cross-stream line through O_1 and the clocks at O_1C_1 are accurately synchronised; consequently the S_1 measure of PQ will be correct, *i.e.* it will agree with the measure of S_2 in which PQ lies. That is, stating it more generally, both worlds agree in cross-stream measurements, because in such

cases their clocks synchronise properly, so there is no ground for suggesting cross-stream contraction or expansion.

But if PQ lies downstream things are different. As PQ is in S_2, we must regard S_2 as the standard, or rest-world, and think from its standpoint; S_1 is moving to the right, but it will serve equally well to think of PQ as moving to the left.

When P arrives at O_1, Q is at A_1 where the clock is slow, so the report of A_1 is disregarded. Q moves on to the left for a time θ in true or S_2 measurement, and arrives at C_1.

If $PQ=l$, $O_1C_1=l-u\theta=\beta(l-u\theta)$ in S_1 units; $\therefore$ the clock at C_1 is late by $\beta(l-u\theta)u/c^2$ of it own seconds; that is, by $\beta^2(l-u\theta)u/c^2$ or $(l-u\theta)u/(c^2-u^2)$ of S_2 seconds.

If then, $\theta=(l-u\theta)u/(c^2-u^2)$ the clock at C_1 will mark zero when Q arrives, and this will be the selected station. We have $\theta=lu/c^2$; $\therefore$ O_1C_1 in S_1 units

$$=\beta(l-u\theta)=l/\beta.$$

That is, while S_2 ascribes to PQ the measure l, S_1 calls it l/β, *i.e.* he measures it as $1/\beta$ of its " true " length.

The relation of this apparent longitudinal contraction of one world as observed by another to the Fitzgerald contraction is worth detailed consideration.

First, the two arise from different causes. The former is due to the differences in synchronisation of clocks. The latter is required (on the aether hypothesis) to explain the apparent equality of light velocity along and across the line of motion. Second, the former is reversible; if S_2 observes S_1 it attributes the same contraction to S_1 as the latter does to itself; the latter is not reversible, for the aether only exists as a hypothetical medium or locus for the transmission of light independent of any material world.

Last, the two are independent, and we cannot get rid of the necessity for the Fitzgerald contraction merely by taking account of the apparent contraction—unless we shut our eyes to the difficulty set forth on p. 361. We can of course say, as the orthodox appear to do, " The aether is unobservable and therefore to us non-existent; it is sufficient to say that light travels 'in the world' —'in any world'—with uniform velocity in all directions; we know nothing more, and need nothing more—everything else is mere speculation and philosophically erroneous." But this does not get over, it merely ignores, the difficulty.

However, leaving these abstruse questions, it may be pointed out that it is a useful exercise to consider two worlds S_1, S_2, both moving through the aether, for simplicity with velocities u_1 and u_2 in the same direction, and see how the fundamental observations (the setting up of the light clock) in S_2 will be regarded by S_1.

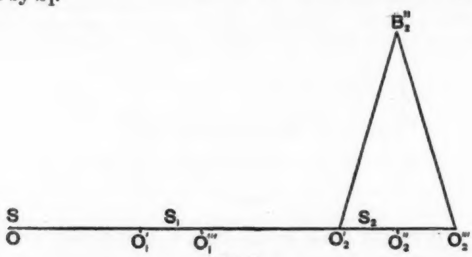

FIG. 5.

Take first S_2's cross-stream observation for fixing his " second "—he emits a flash at O_2' which is reflected at B_2'' and returns to O_2'''. These points at the

various instants in question pass through points in S_1 and the observations are watched and noted by the observers in that world and reported to O_1, which meanwhile itself moves from O_1' to O_1'''.

Let the true or S measure of $O_1'O_2'$ be d, then the S_1 measure of it is $\beta_1 d$, and the clock at O_2' is late by $\beta_1 du_1/c^2$.

Taking $O_2B_2=c$, we have $O_2'O_2''=\beta_2 u_2$, $O_2'B_2''=\beta_2 c$ and $O_2'O_2'''=2\beta_2 u_2$. O_2 takes $2\beta_2$ true seconds to travel this distance and meanwhile O_1 travels $2\beta_2 u_1$;
$\therefore$ $O_1'''O_2'''=d+2\beta_2(u_2-u_1)$ and the S_1 measure of this is $\beta_1\{d+2\beta_2(u_2-u_1)\}$;
$\therefore$ the S_1 clock at O_2''' is late by $\beta_1\{d+2\beta_2(u_2-u_1)\}u_1/c^2$.

Therefore S_1 takes the time between emission and return of the flash as $2\beta_2/\beta_1-2\beta_1\beta_2(u_2-u_1)\,u_1/c^2$, which reduces to $2\beta_1\beta_2(1-u_1u_2/c^2)$; $\therefore$ according to S_1 the S_2 second is $\beta_1\beta_2(1-u_1u_2/c^2)$ of his own seconds.

Also the S_1 measure of the distance S_2 has moved with regard to himself, viz. $(O_1'''O_2'''-O_1'O_2')\times\beta_1$ is $2\beta_1\beta_2(u_2-u_1)$;

$$\therefore \text{ the velocity that } S_1 \text{ ascribes to } S_2 \text{ is } (u_2-u_1)/(1-u_1u_2/c^2).$$

Now take a downstream distance O_2A_2 which S_2 itself calls l, but which, as we have already seen, is only l/β_2 because of the real or Fitzgerald contraction required to mask the effects of motion with regard to S.

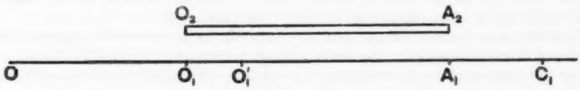

FIG. 6.

Using the same method as before, suppose O_2 passes O_1 at zero; A_2 is then at A_1, where the clock is late, so the A_1 observation is rejected. S_1, S_2 continue to travel to the right and after a further true time θ, A_2 has travelled $u_2\theta$, arriving at C_1, and O_1 has travelled $u_1\theta$ to O_1'; $\therefore$ the true measure of $O_1'C_1$ is

$$l/\beta_2+\overline{u_2-u_1}\theta, \text{ and the } S_1 \text{ measure is } \beta_1(l/\beta_2+\overline{u_2-u_1}\theta);$$

$$\therefore \text{ the clock at } C_1 \text{ is late by } \beta_1(l/\beta_2+\overline{u_2-u_1}\theta)u_1/c^2.$$

This is in S_1 seconds, $\therefore$ in S seconds the clock is late by

$$\beta_1^2(l/\beta_2+\overline{u_2-u_1}\theta)u_1/c^2.$$

Putting this equal to θ to determine C_1 as the point where the clock marks zero when A_2 arrives, we get

$$\theta=lu_1/[c^2\beta_2(1-u_1u_2/c^2)];$$

$\therefore$ in S_1 measure $O_1'C_1$ $=\beta_1(l/\beta_2+\overline{u_2-u_1}\theta)$,

which reduces to $=l/[\beta_1\beta_2(1-u_1u_2/c^2)]$.

S_1 then ascribes to S_2 a longitudinal contraction in the ratio

$$1:\beta_1\beta_2(1-u_1u_2/c^2).$$

If we put $u=(u_2-u_1)/(1-u_1u_2/c^2)$ so that u is the mutual velocity of S_1, S_2 as judged by S_1, and therefore also by S_2, we have

$$\beta^2=1/(1-u^2/c^2)=(1-u_1u_2/c^2)^2/[(1-u_1^2/c^2)(1-u_2^2/c^2)]=\beta_1^2\beta_2^2(1-u_1u_2/c^2)^2.$$

The results, then, which we have obtained by considering in detail the separate motions of S_2, S_1 are the same as those obtained by considering the motion of S_2 relative to S_1, viz. S_1 ascribes to S_2 a velocity u, a "second" which is β times his own and a longitudinal contraction in the ratio $1:\beta$.

We have now established the transformation and have attached a simple and natural meaning to all its details. In certain respects our treatment and interpretation differ from that generally adopted. We have relied throughout not on a mechanical clock, but on the light clock, and have

thereby gained the advantage of perceiving how the clock adjusts itself to changes of world velocity: the "second" lengthens automatically as the velocity increases. We are thereby freed from any need for speculation as to the effect of world-motion on mechanical clocks, and we see precisely and clearly why in changing from one system to another, not dt but βdt is the invariant. Ordinarily this fact appears merely as an algebraic result of the transformation.

The light clock is no doubt unrealisable in practice in its complete form; the distance OB on which it is based would have to be too great for the space at our disposal, at all events on this earth. But as a differential instrument for detecting (or as it proved, failing to detect) difference of light times over equal distances in different directions, it is realised in the apparatus of the Michelson-Morley experiment.

A still more important difference between the usual treatment and that here adopted regards synchronisation. Einstein, for instance (*The Meaning of Relativity*, p. 30), *defines* synchronisation as being effected by the method we have used; that is, he regards it as absolute synchronisation for the world in question. We have preferred to regard the operation as an attempt at synchronisation, the best possible, no doubt, but still imperfect; and we have spoken throughout of the error of synchronisation; clocks being late or fast on the central clock according to their position.

That this is a reasonable as well as a simple point of view appears from the following considerations. S_1 does his best to synchronise his clocks, but he knows not only that other worlds will criticise his results, but also that he will criticise theirs. He knows, in fact, that his system is not perfect—and he knows why, viz. that he is dependent on light signals, and though these are very rapid, they are not instantaneous; if he had any more rapid means of communication he would substitute them for light signals and thereby be surer of an accurate result. He knows further that, although light appears to him to travel with the same velocity in all directions in his own world, this result is only attained by a special arrangement of his methods for measuring distances and time intervals. Light does not really travel in his world with the same speed in all directions, nor with the same speed in other worlds; for though the *measure* of the velocity is the same in all worlds, and in all directions, the velocity itself is not the same, for the units are different. If for instance S_1 regards S_2 as moving with a very high velocity, he will say that in S_2 light travels very slowly, and that the measure is kept constant by adopting a very long second.

The point of view we have adopted leads to a much simpler method of considering, and interpreting, differences of opinion between world and world, as to the time order of events. Suppose that in S_1 two events happen at $x_1y_1z_1$, $x_2y_2z_2$, at the local times t_1, t_2 respectively. S_1 will say that the latter follows or precedes the former in time according as t_2-t_1 is positive or negative.

But another world will challenge his clocks and will say that the one is fast or slow on the other, possibly to such an extent as to reverse the sign when the correction is made. Let us see what are the limits of possible error. The error, real or alleged, depends upon x_2-x_1 and is $(x_2-x_1)u/c^2$.

Suppose that according to S_1 the distance between the two places is r; then the alleged error will be greatest when r lies along the line of motion of S_1 as regarded by the other world; so we will substitute r for x_2-x_1, and the error is ru/c^2.

Further, the alleged error will be greatest when u is greatest. But the greatest values of u of which account need be taken are $\pm c$, for our whole system breaks down for relative velocities higher than c; we have, in fact, no experience of worlds moving with such velocities. The extreme limits of error are therefore $\pm r/c$, in fact, the time that a light signal would take to travel from one place to another.

Now S_1 knows all about this, for ex hypothesi he would apply the same criticism to other worlds, therefore it behoves him to take account of it himself. If then, $t_2 \sim t_1 > r/c$, S_1 will be justified in concluding that the time order he attributes to the events is beyond criticism; but if $t_2 \sim t_1 < r/c$, he can only say, "I think the time order is so and so; but the margin is so fine that I cannot be absolutely certain."

Let us take two illustrations: first the battles of Hastings and Waterloo. A light signal would travel from place to place in about a thousandth of a second, whereas, according to our reckoning, the time interval was 750 years: there is no possibility of any observer reversing our judgement that Hastings happened first.

Distances on the earth are so small that we must go outside it for a practicable case where the time order might be reversed, and a difficulty arises; there are no local clocks, nor local observers to report the times of events; what takes their place?

Happily the answer is easy; if the event is observed by vision from a distance and allowance made for the time of light travel at the normal rate over the distance in question, the time so calculated will be the same as that of a local clock.

Suppose then that when Jupiter and Saturn are at opposite points of their orbits—as far apart as possible—the times of occultation of a satellite of each are noted from the earth, and, allowance being made for the distances, the one precedes the other by one hour. Is the time order beyond doubt?

The time of light-travel from one to the other in these positions is about two hours, so the margin of difference is not sufficient to answer the question with certainty. An observer in stellar space might say *: "Your whole solar system is moving along the line Saturn-Jupiter with high velocity, therefore the light from Jupiter reaches you more rapidly, that from Saturn more slowly, than you think, so your estimate of the interval between the events is wrong."

If the velocity of the solar system were one thousandth of that of light, the error would amount to this fraction of 2 hours, *i.e.* about 7 seconds, and the margin of one hour would be amply sufficient; but by reducing the margin or increasing the supposed velocity it is possible to reverse the sign of $t_2 - t_1$, and so reverse the judgement as to the time order of the events. W. C. FLETCHER.

GLEANINGS FAR AND NEAR.

302. "... the practice which prevails amongst the medical students here [at Edinburgh] of submitting to a private examination by a graduate of the university, before their trials in presence of the professors; this has got the name of 'grinding.' Every college has its 'grinders.'"—*Blackwood*, vol. iv. p. 375 (January, 1818).

303. From a Higher School Certificate Essay:

"... The study of mathematics should be taken in moderation, as there is a limit to everything, and unless this is done, the student is gradually converted from a 'man' into a 'being.' He will become narrow-minded, and will live his own low life apart from the rest of the world."—[*Per* Professor W. P. Milne.]

304. "Having bought Burrow's *Euclid* at Plymouth, and it being the only book I had brought ashore with me, I used to take it to a remote corner of the island, and draw diagrams with a stick on the sand." In this way he mastered the first six books of *Euclid*.—The Rev. J. Newton [Cowper's friend].

* This glib statement while not wholly inaccurate is insufficient, even in the case under consideration; if Jupiter and Saturn are on the same side of the earth, it breaks down altogether, but the proper explanation (by the clocks) holds for both cases.

EXAMPLES OF METHODS OF DRAWING GEOMETRICAL ARABESQUE PATTERNS.

By E. Hanbury Hankin, M.A., Sc.D.

Among the complicated rectilinear patterns employed in Saracenic art there is a group which may be described as "geometrical arabesques," drawn by a curious method which may be of interest to mathematicians.*

A "repeat" of a complicated pattern of this class is shown in Fig. 1. The centre of the repeat is occupied by a sixteen-rayed star. A quarter of such a star occupies each of the four corners of the repeat. In the lower left-hand quarter of the drawing the construction lines are shown, by which this pattern may be easily drawn. These construction lines, shown as single lines, consist of certain polygons. The centre of each side of each polygon is crossed

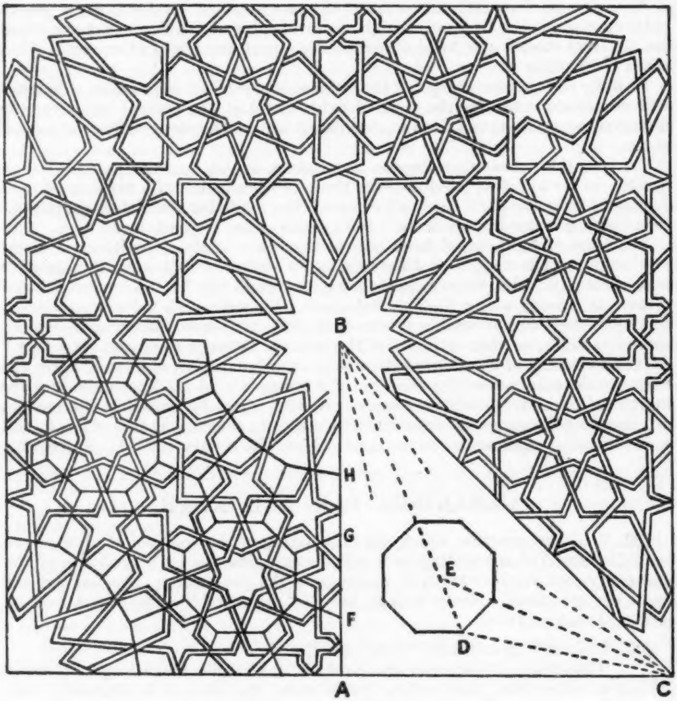

Fig. 1.

* An account of my discovery of this method, by finding some of the original construction lines, was published in a paper "On some Discoveries of the Methods of Design employed in Mahomedan Art" in the *Journal of the Society of Arts*, 1905, vol. liii. p. 461. A more detailed account is on the point of publication in the *Memoirs of the Archaeological Department of the Government of India*.

by two pattern lines. The mode of drawing these pattern lines may be easily seen by inspection. Those that enter the construction octagons are drawn parallel to their radii. Certain of the pattern lines entering other polygons are drawn in line with each other, and so on.

The only difficulty is in drawing the polygons. One has to commence with the octagons. To do so, proceed as follows. The point B is the centre of the repeat. Divide the angle ABC into four equal parts, which are indicated by dotted lines. Do the same with the angle ACB. Two of the dotted lines intersect at E. Then E is the centre of the octagon required. Another intersection occurs at D. Then the line ED is the radius of the octagon. Other polygons are formed by prolonging the radii of the octagons, as shown in the left-hand part of the drawing. The exact size of the 16-gon, which is the basis of the 16-rayed stars of the pattern, does not matter. If GH is marked off equal to AF, then BH, or thereabouts, will be the radius of the 16-gon.

A curious modification of this pattern is shown in Fig. 2. The corners of the repeat are each occupied by quarters of sixteen-rayed stars, but the

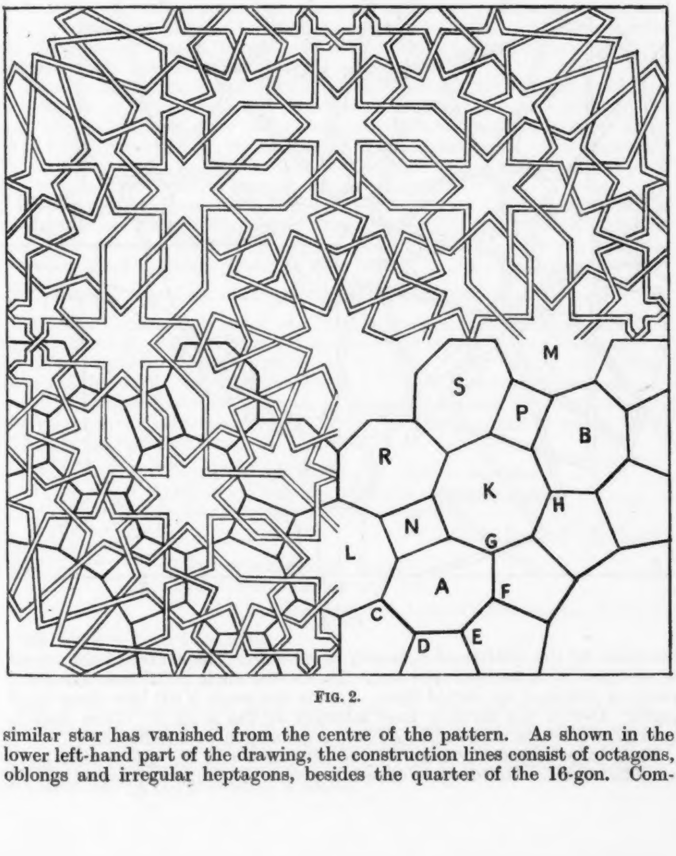

FIG. 2.

similar star has vanished from the centre of the pattern. As shown in the lower left-hand part of the drawing, the construction lines consist of octagons, oblongs and irregular heptagons, besides the quarter of the 16-gon. Com-

mence by drawing the two construction octagons of the same size as and from the same centres as in the previous pattern. But four sides only of each octagon are drawn, as, for instance, *CD*, *DE*, *EF* and *FG*. Then draw a complete octagon of the same size (as *K*), with two of its angles coinciding with the points *G* and *H*. Repeat these complete octagons at *L* and *M*. Join opposite angles of these octagons, thereby forming oblongs *N* and *P*. Thereby irregular heptagons as *A* and *B* have been formed as residual spaces. Repeat these irregular heptagons at *R* and *S*. Having completed these construction lines the pattern may be drawn by the same method as before, and needs no further description.

A singular example of the ingenuity of the Arabian craftsmen is shown in Fig. 3. The repeat of this pattern is a square. Each of the four corners is

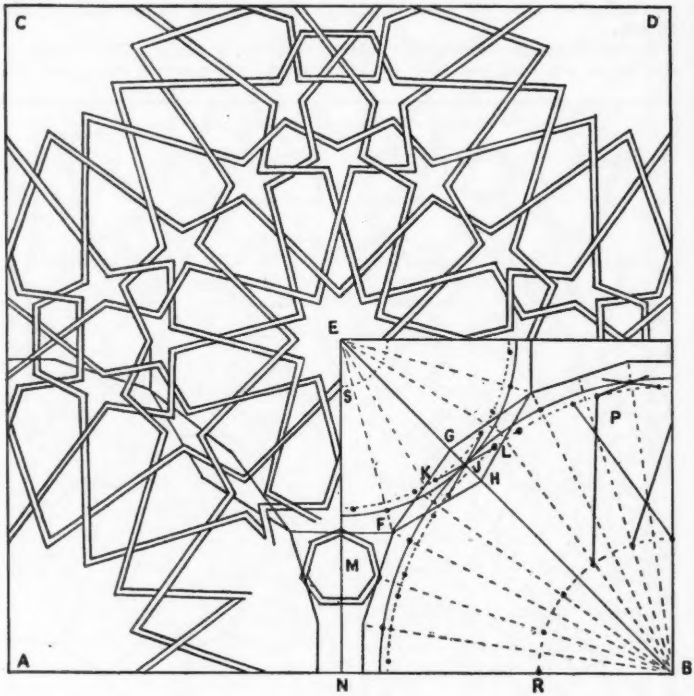

FIG. 3.

occupied by the quarter of a twenty-rayed star. The centre of the repeat is occupied by a twelve-rayed star. Divide the angle *NBE* into five equal parts as indicated by dotted lines. Divide the angle *NEB* into three equal parts. Two of the dividing lines intersect at the point *F*. Then draw a quarter of a 20-gon with radius *BF* and a dodecagon with radius *EF*.

These large polygons overlap by two sides at *G* and *H*. At *M* they leave a residual space. In this space draw a heptagon as regular as possible, having

three of its angles touching centres of neighbouring sides of polygons. The heptagon thus drawn becomes part of the pattern. Contrary to the usual practice, no further use is made of the sides of the polygons in drawing the rest of the pattern. From centres E and B, two circles are drawn which touch each other at a point midway between G and H. These circles are shown as continued lines. Pairs of pattern lines are drawn through each point where these circles intersect a radius. Inside these circles slightly smaller circles are drawn, whose size has to be guessed, and which are drawn as dotted lines. Points are marked, as indicated by dots, on each side of the position where these circles cut interradii. Pattern lines meet in these dots as indicated at P. The exact distance of these dots from the interradii does not matter. Smaller circles at R and S are also required. Their exact size is unimportant, and is a matter of guess-work. The point to be aimed at in making these guesses is that the pattern spaces should be as regular as possible. If radial symmetry is impossible, at least a bilateral symmetry should be aimed at.

A feature of all these patterns is that two lines cross at a point, but that a point is never a meeting-place for several lines. Hence each line, after aiding in limiting one pattern space, runs on to aid in limiting another pattern space. Thus the different parts of the pattern are interdependent. This appearance of interdependence may be increased, with addition to the aesthetic effect, by interlacing the lines as in the examples here drawn.

Usually, as in Figs. 1 and 3, each pattern line starting from any point on the margin of the repeat may be followed in a zigzag course through the pattern till it reaches some other point on the margin. But in some cases, as in Fig. 2, some of the pattern lines mark out closed areas. It is impossible for me to see from the construction lines whether or not such enclosures will be found in the completed pattern.

Of the patterns here described, Figs. 1 and 2 are from a dome in the Alhambra (see *Moorish Remains in Spain; the Alhambra*, by A. E. Calvert), and Fig. 3 is copied from Bourgoin, *Les Elements de l'Art Arabe; le Trait des Entrelacs* (Paris: Firmin-Didot et Cie, 1879). E. Hanbury Hankin.

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305. The Cocker of America was Nathan Daboll. A catalogue published by Mr. Barnard, of Tunbridge Wells, contained several large 12mo copies of *The New England Almanac and Gentlemen and Ladies' Diary* by Nathan Daboll (1798, 1801, 1803, 1805). Printed at New London. *Hutchins' Improved* or *Father Hutchins' Revived*, by Father Abraham Hutchins, Mathematician, was a rival almanac, large 12mo, New York, copies of which, 1794-1806, were in the same list.

306.
When Long and Short give place to angles,—
When stern Mathesis makes it treason
To like a rhyme or scorn a reason.

—*Surly Hall*, W. M. Praed.

307.
Life has other hopes than Cocker's,
Other joys than tare and tret.

—"The Invocation," Aytoun's *Bon Gaultier Ballads*.

308. From a Book Catalogue:
Curious.—The Conical Adventures of Beau Ogleby. 1829.
[Is this description hyperbolic? or was he a refractory Beau?]

309. Heine describes Dr. Saul Ascher as "a personified straight line." This may have lingered in the memory of Mrs. Harriet Beecher Stowe, whose Dr. Drake "is a tall, rectangular, perpendicular sort of a body, as stiff as a poker, and enunciates his prescriptions very much as though he were delivering a discourse on the doctrine of election."

MATHEMATICAL LABORATORY: ITS SCOPE & FUNCTION.*

By Prof. H. Levy, D.Sc.

The function of the mathematical laboratory can best be understood in the light of the experience of many mathematical teachers. Anyone who has devoted even a comparatively short time to the mathematical training of workers in any branch of applied science must have had the experience that many past students return at frequent intervals for advice and consultation. The difficulties with which they are confronted appear invariably to be associated with the application in more or less concrete form of the mathematical principles and methods which they have learned. I find, as a matter of actual experience, that the complaint, when it is voiced, concerns itself not so much with the width of the mathematical field covered during training as the inability to know when to apply this or that mathematical method, how to attain the so-called solution, and how, when the mathematical solution has been obtained, to derive from it the desired information with reasonable expenditure of labour. It is partly this gap that the mathematical laboratory seeks to fill. An engineer confronted with a problem in heat conduction will not be satisfied should you provide him with an expression for the distribution in temperature in terms of a series of Bessel functions, unless he has at hand the means of immediately calculating to so many decimal places the values of these functions, either from a set of tables or by some direct mechanical process which he can apply. The form of solution provided may be to him more difficult to interpret than the original problem. It is partly because in the past mathematical teachers have been content to regard a problem as solved when, somehow or other, an analytical solution has been obtained, or to regard a problem as insoluble when such a solution cannot be obtained, that the failure exists on the part of the students of that training to utilise the mathematical weapons at their disposal, to modify them according to circumstances, and to interpret their results in actual cases.

Now, it is important in the first place to recognise that any problem with a concrete application is subject to certain definite practical limitations. The various physical constants specifying the nature of materials are known only to a limited degree of accuracy. The idealisation of geometrical conditions and restraints that are assumed in the theoretical problems are only partially realised in practice. A beam under loading, for example, held by pin-jointed supports at both ends, only fulfils approximately the conditions assumed in the ordinary theoretical treatment. It follows frequently that what the practical experimenter requires from the mathematician is not the solution of the mathematician's conception of a problem correct to ten places of decimals, but an answer of such limited accuracy as will correspond to the inaccuracy of the physical constants involved, and the approximateness of the physical conditions assumed.

From this point of view the function of the mathematical laboratory is to devise such convenient methods of analysis and calculation as will provide the solution of the problem to the required accuracy, and this for most practical purposes—engineering, chemical, physical—would correspond to three or four places of decimals.

But the function of the laboratory is not limited merely by this purely utilitarian aspect. I have learned by experience in my laboratory at the Imperial College that the graphical and arithmetical methods we provide usually effect a separation of the students into two groups, corresponding in some measure to their capacity for visualising abstract ideas. Students mentally unable, apparently, to grasp either the necessity for an Existence Theorem or unable to follow a piece of pure analysis leading towards the solution, say of a differential equation, will have no difficulty whatsoever in

* A paper read at the Annual Meeting, 5th January, 1925.

arriving at the necessity for such analysis once they have followed in pictorial form at first hand the various stages in the successive approximations by a curve to the solution of such an equation.

There is no doubt whatsoever that graphical and arithmetical methods, especially the former, are a potent means of quickening and stimulating the mathematical interest of students, who would otherwise be classified as stupid; and many of these students by this process have been led back to the more pure, and possibly the more fundamental, branches of mathematics by the insight they have acquired from their training in the mathematical laboratory.

Let me take a particular illustration. Let us suppose we are concerned with the solution of a differential equation of the second order, derived as a representation of the physical laws governing some specific problem. Under the circumstances, it is very exceptional if the investigator is unable to give a rough guess—it may be very rough—to the answer. The student begins with this rough guess as a first approximation expressed, say, as a curve in the plane xy. The values of x and y taken from this curve when inserted in the differential equation wherever these quantities appear will enable an approximate value of y'' to be calculated. Plotting this on a base x and integrating with a planimeter, or otherwise, a second approximation to y may be derived. By repetition of this process successive approximations may be obtained. A few cases of this nature worked out in detail are sufficient to raise in the mind of any student, no matter how apparently unmathematical he may be, the following questions:

(1) How do I know that this sequence of approximations will converge to some definite curve?
(2) How do I know that it will converge to the solution of the problem?
(3) Does the rapidity of its convergence depend on the range? and
(4) Under what circumstances will the rapidity of its convergence be independent of the range?

We see here that the process naturally suggests to the student most of the essential questions connected with the meaning of a sequence, the meaning of convergence and the distinction between ordinary and uniform convergence.

In actual fact, I find that students whose interests are initially dead to these ideas suddenly become alive to their importance, and of their own accord turn to the mathematical analysis suggested by their graphical experience. It is no exaggeration to say that there is scarcely a field of mathematical investigation which cannot be opened up by this method to the interest of the student otherwise unmoved by it.

Let me give briefly the essentials of a course in such a laboratory. The student would begin with problems designed to provide experience in direct computation involving the compilation and use of sets of tables of different types of functions, the use of the slide-rule and multiplicating machine. At this stage, and for this purpose, the properties of series, their convergence, the rapidity of their convergence, their uniformity or otherwise, may be brought home to the student by direct processes of computation. By graphing a series, such as, for example, that of the Bessel function, whose terms involve a variable, the conception of an infinite series, of a function with an infinite number of zeros, and numerous special properties may be expounded. If at this stage the use of the planimeter be introduced the student may convince himself by methods of direct integration that these series satisfy certain differential equations. Thus, the initial fear of such functions as those of Bessel and Legendre is eliminated by the familiarity with their properties which the student thus acquires in tabulating and analysing them.

A little treatment of nomography and a detailed study of the graphing of functions extending as far as the use of Newton's diagrams is then introduced. The next stage would deal with simple propositions in finite differences leading up to questions of interpolation and extrapolation, and the use of these ormulae for numerical differentiation and integration. At this stage also,

graphical methods for effecting these processes can, with advantage, be introduced in order to impress on the students the advantages and disadvantages of both. The next stage should deal with the representation of a function, graphically or arithmetically defined, by Fourier series. It should here be possible to deal in detailed numerical form with problems of conduction of heat and electricity.

The next stage deals with the treatment of differential equations. The formula developed in connection with the finite differences can be immediately utilised in numerical solution, while the methods of successive approximation already outlined are applicable when the accuracy of the required solution is limited within the scope already indicated. In this treatment no distinction need be made between linear and non-linear equations, and it has already been explained how a discussion of these methods of solution raise problems of vital importance in pure mathematics.

A similar process may be applied with considerable success to the treatment of partial differential equations. Differential co-efficients may be represented approximately in terms of small differences, and the equation then determines the law governing the variation of the function with these small differences. It is in the handling of such a problem by this method, in the casting of the arithmetical process into manageable form, that a sound training in processes of computation makes itself apparent. We have found that many of the methods developed by Runge can be applied with considerable success to the determination in arithmetical form of the solution of physical problems in heat and electrical flow and in the torsion of cylinders. Graphical methods may be applied to similar types of problems, either by the methods suggested by Richardson and others of starting from a rough guess and amending the solution by some definite process depending on the nature of the equation, or as in the equation $\nabla^2 V=0$, by a synthetic process of building up an approximation to a desired solution by the super-position of elementary types. This latter process enables the student to acquire a real working familiarity with Green's functions and the process of equivalent strata by building up solutions from a distribution of sources and sinks over the boundaries.

If it is futile to expect that average students of the purely experimental sciences should ever acquire such proficiency in the use of mathematical weapons as will enable them to become real instruments of discovery in their hands, it is at any rate possible to provide them with a knowledge of mathematics sufficient for the analysis of their experimental observations. No mathematical laboratory which hopes to develop the functional side of mathematics can be complete unless it provides facilities for the discussion of laws of error, curve fitting, and the judgement of the validity and the consistency of experimental results by direct laboratory practice. No such student, once he has acquired a real working knowledge of these methods in his early days, should fail to apply them in later work, always provided that the matter dealt with in the mathematical laboratory is really directly correlated with the realities he will be called upon later to face.

But a mathematical laboratory ought to be more than this. In course of time it must open up lines of enquiry into fields of application at present closed to direct analysis. With the application of science and the scientific method to diverse fields of industry, with the greater and greater introduction of scientific workers into industrial research associations, with the development of Government Research Laboratories on a large scale, mathematics in the service of science will be called upon more and more to functionalise itself. It will be required to develop mathematical methods specially adapted to the investigation of the new kinds of scientific problems opened up. Traditional restrictions of purely logical interest will require to be swept aside, and any form of experiment will require to be allowed side by side with the experimental step by step mathematical process.

Imperial College of Science, London. H. Levy.

SOME THEOREMS IN GEOMETRY AND SOME SUGGESTIONS.

By R. T. Robinson, M.A.

(*Continued from p.* 282.)

(9) Take a triangle and any point S in its plane. Reciprocating with respect to S we get another triangle ABC, say.

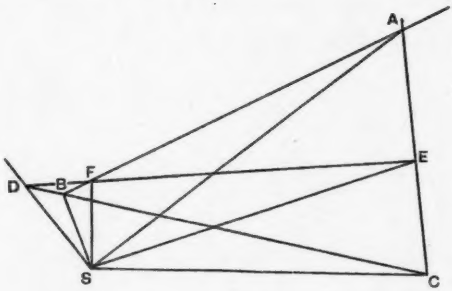

FIG. 12.

If SD	is drawn $\perp^r$ to	SA	to meet	BC in D,	
and SE	,, ,,	SB	,,	CA in E,	
and SF	,, ,,	SC	,,	AB in F.	

then $D \,.\, E \,.\, F$ are collinear and the line DEF is the reciprocal of the orthocentre of the original triangle. This st. line does not seem to have received the attention it deserves. Until it receives a better name, the writer will call it the ortho-line of S with respect to the $\triangle ABC$.

Given the ortho-line DEF and a $\triangle ABC$, the point S can be found by describing circles on $AD \,.\, BE \,.\, CF$ as diameters. These circles intersect in the points S and S_1, real, coincident or imaginary, *i.e.* the two points S and S_1 have the same ortho-line with respect to the $\triangle ABC$.

(10) If O is the ortho-centre of the $\triangle ABC$, $O \,.\, S$ and S_1 are collinear and

$$OS \,.\, OS_1 = -4R^2 \cos A \cos B \cos C,$$

i.e. S and S_1 are inverse points with respect to the self-polar circle of the $\triangle ABC$.

(11) The nine-points circle of the $\triangle ABC$ is the inverse of the circle ABC with respect to the self-polar circle. Therefore if S lies on the circle ABC, S_1 will lie on the nine-points circle.

Also, if S lies on the self-polar circle, S and S_1 will coincide.

(12) The following Theorems are important.

(1) If S lies on the circumcircle of the $\triangle ABC$, the ortho-line DEF passes through the centre of the circle.

(2) The locus of the Frégier points of S with respect to all conics passing through $A.\ B.\ C.\ S$ is the ortho-line of S.

(3) The ortho-line of S with respect to the four triangles formed by 3 of the points $ABCD$ meet in a point—the Frégier point of S for the conic $A.B.C.D.S$.

(4) If the $\triangle ABC$ circumscribes a conic and if S lies on the director circle of the conic, the ortho-line of S with respect to the $\triangle ABC$ touches the conic.

(5) The ortho-line of a focus S of a conic to which the $\triangle ABC$ is self-conjugate is the corresponding directrix.

(13) Generalisation of the Theorem in (9).

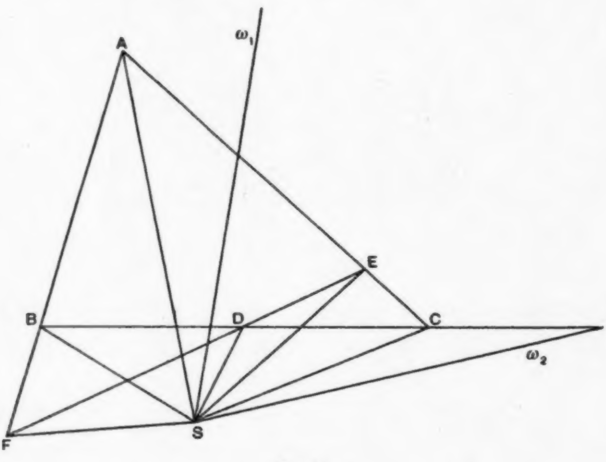

FIG. 13.

Given a $\triangle ABC$ and two base lines $S\omega_1$, $S\omega_2$ where ω_1, and ω_2 are any points on the lines.

Draw SD $\perp^r(\omega_1\omega_2)$ to SA *i.e.* so that $S(A\omega_1 D\omega_2) = -1$.

" SE " " SB " " $S(B\omega_1 E\omega_2) = -1$.

" SF " " SC " " $S(F\omega_1 C\omega_2) = -1$.

Then DEF are collinear and DEF is the ortho-line $(\omega_1\omega_2)$ of S with respect to the $\triangle ABC$.

This is the reciprocal of Desargues' Theorem, for in the quadrilateral $ABDE$, $SA.SD : SB.SE : SC.SF$ are the pair-lines of an involution pencil of which $S\omega_1$, $S\omega_2$ are the double lines. Further, this reciprocal theorem can be stated in another way as follows:

In a quadrilateral $ABDE$ each side is the ortho-line $(\omega_1\omega_2)$ of a point S with respect to the triangle formed by the other three sides.

Or, if $S\omega_1$, $S\omega_2$ are conjugate lines with respect to a conic and if the $\triangle ABC$ circumscribes the conic, tne ortho-line $(\omega_1\omega_2)$ of S with respect to the $\triangle ACB$ is also a tangent to the conic.

(14) We have stated that if S lies on the circumcircle of a $\triangle ACB$, the ortho-line of S with respect to the $\triangle ABC$ passes thro' the centre of the circle.

The generalised theorem is this:

If any conic is drawn through the four points $ABCS$ and if the base lines $S\omega_1 S\omega_2$ meet the conic in P_1P_2, the ortho-line $(\omega_1\omega_2)$ of S with respect to the $\triangle ABC$ passes through T the pole of P_1P_2.

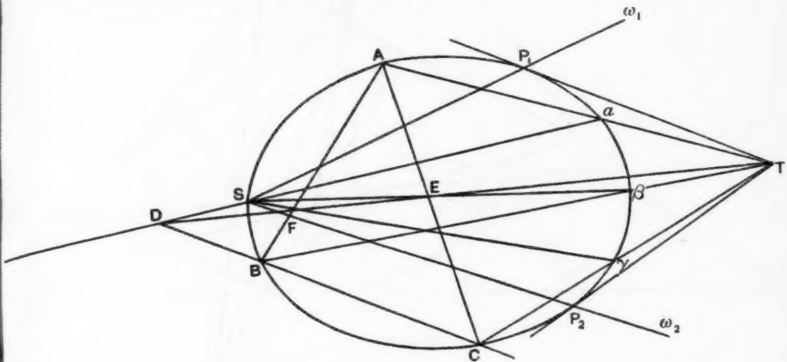

FIG. 14.

Join AT meeting the conic again in a, then since $S(A\omega_1 a\omega_2) = -1$,

Sa is $\perp^r$ $(\omega_1\omega_2)$ to SA : let sa meet BC in D.

Similarly if BT meets the conic again in β, $S\beta$ is $\perp^r$ $(\omega_1\omega_2)$ to SB.

Let $S\beta$ meet AC in E,

and if CT meets the conic again in γ, $S\gamma$ is $\perp^r$ is $\perp_r$ $(\omega_1\omega_2)$ to SC.

Let $S\gamma$ meet AB in F.

Then $DEFT$ are collinear, *i.e.* the ortho-line $(\omega_1\omega_2)$ of S with respect to the $\triangle ABC$ passes thro' T.

(15) If a hexagon $SABCa\beta$ is inscribed in a conic, every Pascal line of the hexagon is the ortho-line $(\omega_1\omega_2)$ of one of the six points, S, say, with respect to the triangle formed by three other points ABC, say, the base lines $S\omega_1$, $S\omega_2$ being the st. lines joining S to the points of contact of the tangents to the conic from T, the point of intersection of the lines joining A and B to the two remaining points a, β.

Further the theorem in (14) shows that there are 4 fixed points on every Pascal line, *i.e.* the point T and the feet DEF of the $\perp^{rs}$ $(\omega_1\omega_2)$ from S on the sides of the $\triangle ABC$.

(16) If the point T is inside the conic, the base lines $S\omega_1$ $S\omega_2$ are imaginary, but the result is the same.

(17) If 5 points $Aa B\beta S$ on a conic are given, then for every st. line drawn through B, say, this construction enables us to find 2 other points on the conic, *i.e.* the point C where the st. line through B meets the conic again and the point γ where CT meets the conic again.

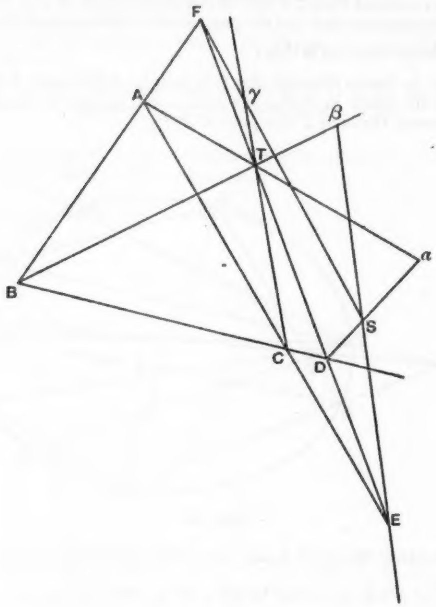

FIG. 15.

The steps of the construction are these:

(1) Draw any st. line thro' B meeting Sa in D.
Then TD is the ortho-line $(\omega_1\omega_2)$ of S with respect to the $\triangle ABC$, whose vertex C is to be found.

(2) Join $S\beta$ meeting TD in E.
Then E lies on AC, *i.e.* C lies on AE, but C lies on BD; $\therefore$ C is the intersection of AE and BD.

(3) Join AE meeting BD in C.

(4) Let TD meet AB in F: join SF meeting CT in γ, then γ is also on the conic for $S\gamma$ is $\perp^r (\omega_1\omega_2)$ to CS.

(18) The reciprocal of (14) is: If *abc* are tangents to a conic and '*s*' is any other tangent and if any two points $\omega_1\omega_2$ on the tangent '*s*' are taken as base points, the ortho-centre $(\omega_1\omega_2)$ of the $\triangle abc$ lies on the polar '*t*' which joins the points of contact, the two other tangents to the conic from $\omega_1\omega_2$.

The tangents $a, a, P_1\omega_1, P_2\omega_2$ are cut by the tangent '*s*' in a harmonic range, *i.e.* $[(sa)\omega_1\ (sa)\omega_2] = -1$;

$\therefore$ the st. line $(bc)(sa)$ or d is $\perp^r\ \omega_1\omega_2$ to the st. line '*a*.'

Similarly

the st. line $(ca)(s\beta)$ or e is $\perp^r$ $\omega_1\omega_2$ to the st. line 'b.'

and the st. line $(ab)(s\gamma)$ or f is $\perp^r$ $\omega_1\omega_2$ to the st. line 'c,'

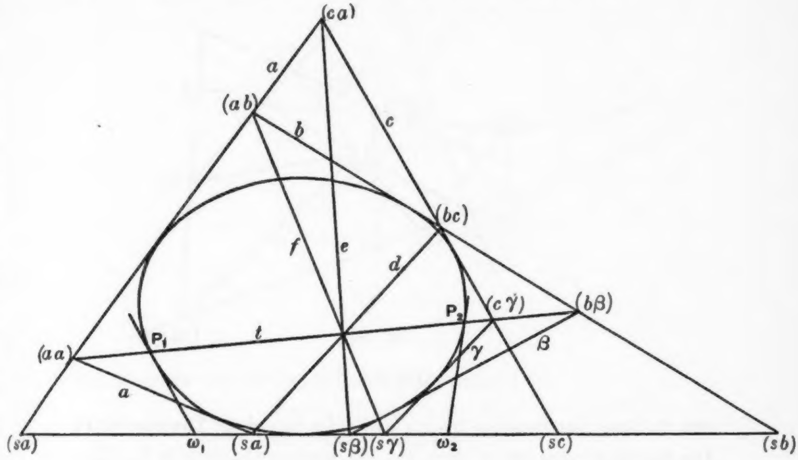

FIG. 16.

the point of intersection of these st. lines is the ortho-centre $(\omega_1\omega_2)$ of the $\triangle abc$ and from the reciprocal of (14) the ortho-centre $(\omega_1\omega_2)$ lies on 't.'

(19) If a hexagon formed by the st. lines S, a, b, c, α, β circumscribes a conic, every Brianchon point of the hexagon is the ortho-centre $(\omega_1\omega_2)$ of the triangle formed by three of the tangents a, b, c, say, the base points being the points of intersection with 'S,' say, of the tangents to the conic at the points where it is met by 't' the st. line joining the points of intersection of $a\alpha$ and $b\beta$.

Further the theorem shows that there are four fixed st. lines passing through every Brianchon point, *i.e.* the line 't' and the $\perp^{rs}(\omega_1\omega_2)$ d, e, f from the vertices on to the sides of the $\triangle abc$.

(20) The Theorem in (18) is the generalisation of the Theorem that the ortho-centre of a triangle circumscribing a parabola lies on the directrix.

(21) Given 5 tangents to a conic, the Theorem in (18) enables us, if we start with any one point on a tangent 'b' say, to draw 2 other tangents.

Let the 5 given tangents be a, α, b, β, s.

Take any point (bc) on b. To draw 2 other tangents one of which is 'c.'

Reciprocate the Problem in (17). The steps are these :

(1) Join $(a\alpha)$ and $(b\beta)$, call this line 't.'

(2) Join $(s\alpha)$ to any point (bc) on 'b'; call this line 'd' and let it cut 't' in O.

(3) Join O and $(s\beta)$. This line 'e' meets a in (ca): thus 'c,' being the line joining (ca) and (cb), is found.

(4) Join (ab) to O meeting 's' in (sf). The line joining (ct) and (sf) is another tangent.

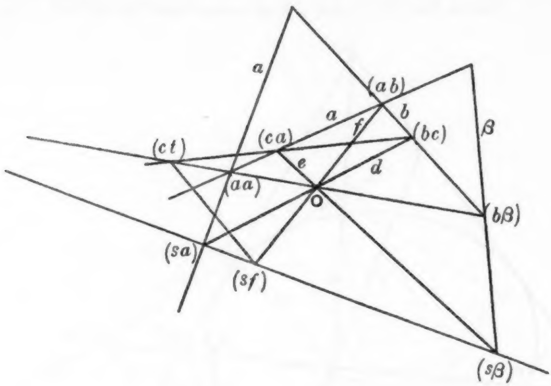

FIG. 17.

(22) To show that Simson's line is a particular case of the Theorem in (14

The base lines being any st. lines $S\omega_1$, $S\omega_2$ meeting the circle in P_1, P_2.

If SD is $\perp^r$ $(\omega_1\omega_2)$ to SA, *ie.*, if $S(A\omega_2D\omega_1) = -1$.

and SE „ „ SB „ $S(E\omega_2B\omega_1) = -1$

and SF „ „ SC „ $S(C\omega_2F\omega_1) = -1$,

then DEF are collinear and DEF passes through the point of intersection of the tangents to the circle at P_1P_2.

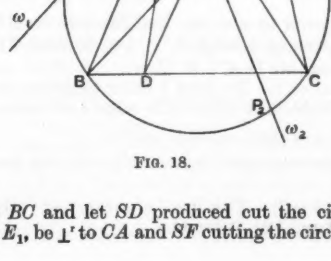

FIG. 18.

Let SD be $\perp^r$ to BC and let SD produced cut the circle in D_1; let SE cutting the circle in E_1, be $\perp^r$ to CA and SF cutting the circle in F_1 be $\perp^r$ to AB.

It can be proved that if P_2 is the mid-point of the arc CF_1, it is also the mid-point of the arcs AD_1 and BE_1. The complementary arcs have a common mid-point P_1 and P_1P_2 is a diameter of the circle; $\therefore$ SP_1 and SP_2 are $\perp^r$.

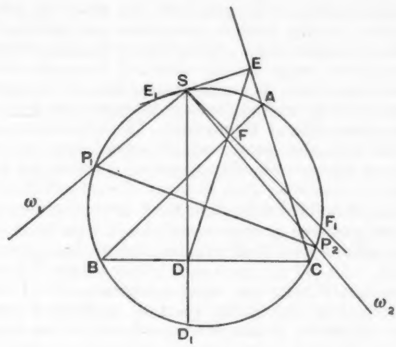

FIG. 19.

This being the case SD and SA are equally inclined to

$$S\omega_2, \textit{ i.e. } S(A\omega_2 D\omega_1) = -1.$$

$$\textit{i.e. } SA \text{ is } \perp^r (\omega_1\omega_2) \text{ to } SD.$$

Similarly SB is $\perp^r (\omega_1\omega_2)$ to SE and SF is $\perp^r (\omega_1\omega_2)$ to SC; $\therefore$ D, E, F are collinear and DEF passes through the point of intersection of the tangents at P_1P_2, *i.e.* DEF is $\perp^r$ to P_1P_2.

(23) The following particular cases are of interest for the conic in general:

(1) when one of the base-points $\omega_1\omega_2$ is at infinity,

(2) when $\omega_1\omega_2$ are conjugate points with respect to a conic,

(3) when the base lines $S\omega_1$, $S\omega_2$ are $\perp^r$,

(4) when the base lines $S\omega_1$, $S\omega_2$ are conjugate lines with respect to a conic.

R. T. ROBINSON.

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310. Agassiz was asked to give his knowledge and genius to a great pecuniary undertaking which would have made him a fortune, but replied: "I have no time to make money"!

311. Cardan ranked Apollonius the seventh of all who have lived.

312. It's a question that would puzzle an arithmetician . . . whether the Bible saves more souls in Westminster Abbey or damns more in Westminster Hall.—(Valentine, *loq.*) Congreve's *Love for Love*, iv. 2.

313. Where's your oppositions, your trines, and your quadrates? What did your Cardan and your Ptolemy tell you? Your Messahalah and your Longomontanus, your harmony of chiromancy with astrology?—(Sir Sampson Legend *loq.*) Congreve's *Love for Love*, iv. 3.

314. Love hates to centre in a point assigned,
But runs with joy the circle of the mind.

—(*Song*). Congreve's *Love for Love*, iv. 3.

MATHEMATICAL CLUBS IN SCHOOLS.[1]

By Miss I. M. Brown, B.A.

I think that everyone here will agree with me that we, who try to teach mathematics, have a heavily loaded curriculum and we are apt to become the slaves of public examinations. According to the age of our pupils, Intermediate or Matriculation hangs like the sword of Damocles above us, and we dare not be led far astray from the mark-gaining part of our work.

Personally I have always wished that my interests had led me either to the classical or the modern side of Education. Their beauties are *so* apparent, and somehow girls, and also people whose school-days are over, are often obstinately surprised that one should suggest an intrinsic joy in mathematics. Of course, one has to remember that to many the subject called mathematics, of such width as is denoted by its derivation from the Greek μανθάνω: to learn by inquiry, to perceive or understand, has for its limit on the algebraic side, quadratic equations and their graphs, and on the geometrical side the VI. Book of Euclid. An old girl once said, loftily to me, "You know, I never *could* do mathematics, I have too much imagination." I happened to remember that very girl as one of the specially pigheaded ones over square roots of negative quantities, a tangent considered as the limit of a secant, and a secant which cuts a circle or curve in imaginary points.

Now the obvious function of a Mathematical Club is to stimulate the interest which *may* have been aroused in the class-toom, or if possible, to awaken an interest hitherto dormant. And I think we should naturally turn first to the History of Mathematics. But for school work, you are at once confronted with a difficulty.

You may pick up a 400-page history, but how much of it will refer to mathematics within the limits I have mentioned ?

There is Ahmes (1400 B.C.) and his *Directions for Knowing all Dark Things.*

Thales (640 B.C.), one of the Seven Wise Men, is a friend to the beginners in Geometry.

Pythagoras (580-500 B.C.), with his extraordinarily wide and human interests can be made an arresting figure.

There is Plato (429-348 B.C.) and his Delian Problem to duplicate the Cube.

Archimedes (287 B.C.) comes down to earth with his circles, sphere, cylinder and irrigation works ; and meets the schoolgirl over the ratio π.

But Apollonius (260 B.C.) and his conic sections. Hipparchus (160 B.C.), of geographical fame, with his trigonometry. Ptolemy (second century A.D.). Diophantus and his problems (fourth century A.D.). Theon and Hypatia . . . are all a long way beyond our matriculation candidate.

Brahmagupta the Hindoo (born 598 A.D. . . . 650) may fix ambition. He certainly stated his problems in striking language. There is this famous one of his : "Two anchorites lived at the top of a cliff of height h, whose base was distant mh from a neighbouring village. One descended the cliff and walked to the village, the other flew up a height x and then flew in a straight line to the village. The distance traversed by each was the same. Find x."

This and other problems, he says, "were proposed simply for pleasure. As the sun eclipses the stars by his brilliancy, so the man of knowledge will eclipse the fame of others in assemblies of the people if he proposes algebraic problems and still more if he solves them."

Then Bhaskara (born 1114 A.D. . . .1140), wins the heart of schoolgirls by solving $x^2-45x=250$; the answers, he says, are 50 and -5, "but the second value is in this case not to be taken, for it is inadequate ; people do not approve of negative roots."

[1] A paper read to the London Branch, by Miss I. M. Brown, Haberdashers' Aske's School, Acton, on October 11th, 1924.

Bhaskara, as you know, was an astrologer and knew that his daughter's marriage would spell disaster to himself, he therefore kept her shut up (how it reminds us of Greek and other legends), and as some sort of consolation, he addressed his mathematical works to her : "Lovely and dear Lilavati, whose eyes are like the fawn's, tell me what are the numbers resulting from 135 multiplied by 12 ? If thou be skilled in multiplication . . . tell me, auspicious damsel, what is the quotient of the product when divided by the same multiplier ?"

But he has a prettier problem than this for "his lovely Lilavati."

"The square root of half the number of a swarm of bees is gone to a shrub of jasmin ; and so are $\frac{8}{9}$ of the whole swarm : a female is buzzing to one remaining male that is humming within a lotus in which he is confined, having been allured to it by its fragrance by night ; say, lovely woman, the number of the bees."

After a very long gap we reach Napier (1550-1617) ; although we now use logarithms pretty soon, it is almost impossible to present Napier to the girls, since his approach to logarithms was so difficult without the indices notation, which was not then invented.

Descartes (1596-1650), merely lends his name to Fourth form graphs.

Pascal (1623-1667), is looked upon by girls as entirely unnatural, though if you reach the Binomial Theorem, they think his Triangle very wonderful.

And that is the sum total and we have only reached the 1600's—and that very scantily. So much for the History of Mathematics.

Suppose then that you ask the members of the Club to "propose problems for pleasure." You very likely have brought you the dullest puzzles that some magazine or an enterprising penny paper can produce ; and I for one hope that the genius of Mathematics forgives the indignity.

In spite of the difficulties, however, I am quite sure that an added interest in Mathematics may be fostered. I am now coming to facts and I propose to tell you of the work actually done by a Club.

We stated that the object of the new club was : "To show the fascination of Mathematics and to explore the byeways." Nearly a hundred girls asked for membership ; we reduced them to forty, knowing that if *we* had not done so, the reduction in numbers would later have been automatic.

We began by a fair amount of graphical work ; girls with a passion for arrangement drew small groups of parabolæ of varying latera recta, and in different positions, showing the range from $y^2=0$ to $y^2=\infty$ —and similar work on ellipses and hyperbolæ. Cleverer girls drew cycloids (the Helen of Geometers is rather appealing), spirals and polar equations such as the cardioid ($r=a(1+\cos\theta)$) and lemniscate. The rather dull girls enjoyed drawing curves of pursuit and Boule curves.

Meanwhile some of the VI. form were enjoying Dr. Abbott's *Flatland*,[1] and they offered to read a paper on the book to the rest of the Club. These readings gave rise to great discussions on the difficulties and restrictions of life in Flatland and Lineland. A great many diagrams were drawn ; amongst others, an enormous plan of a town in Flatland was produced. It was hideous and showed neither imagination nor skill, but the maker was very proud of it and incidentally she had pressed into her service two smaller sisters and they grew up with a zeal for accuracy in measured drawing.

Meanwhile from scraps of overheard conversation girls in the middle school became interested in Flatland and Geometry classes eagerly made wonderfully accurate illustrations at home.

We openly laughed at the Flatlanders for their refusal to believe in any mathematical possibility which they could not actually see or touch. Then when in the ordinary school routine I came across a series of arithmetical quantities going on for ever, and yet their sum was asked for ; when the girls

[1] Little, Brome & Co., Boston.

drew curves which at their present rate of approach would obviously cut an asymptote on the very page; when a girl first solved a quadratic equation having imaginary roots and drew its graph showing the parabola cutting the axis in imagination only; when I introduced $\sqrt{3}-1$ and $\sqrt{3}+1$ as factors of the hitherto prime number 2 . . . then—I referred to the narrow-minded Flatlanders and begged the girls not to fall into their self-satisfied errors—but to use their imagination even in a subject known as " a sober scientific reasoning."

In our next session one group of girls compiled a manuscript or rather typescript book of mathematical puzzles and anecdotes. Another group searched encyclopædias and many books for stories of mathematicians and they produced a volume of Lives of Great Mathematicians. A third group of girls wrote a history of Arithmetic from Babylonian times to the present day paying especial attention to the simplification of notations and quickening of methods. Each book was illustrated.

Towards the end of that session we held an " open " meeting, to which the whole Upper School might come. This was our programme:

1. Some old methods of Arithmetic, scratch and galley division, gingerbread multiplication.
2. The same sums by ordinary methods.
3. Story of Napier, and the use of his rods.
4. Multiplication, root finding, compound interest by logarithms (at that time logs. were not in the metric syllabus and were to most merely mysterious).
5. Percentages read from a ruler and some slide rule calculations.
6. A short mathematical story dealing with elementary algebra.

By this time we had a fair amount of collected stories, puzzles and so forth; so in the following session we made a missionary effort to bring light to the darkened minds of the school at large. Every week on the notice board of each upper school class-room we posted a short story—*e.g.* one about Babylonian notation and the introduction of our minutes and seconds; one on the origin of units such as the furlong, yard, metre; stories of Archimedes and Pythagoras; the Delian Problem.

A fresh idea was suggested for our next year's work; and proceeding on the same lines as before we had a series of papers read and illustrations drawn, showing constellations of stars and their legends.

An " open " meeting was now proposed for the Middle School. At once the absurdity of trying to *entertain* 11 and 12 year olds with a mathematical programme was pointed out. However, if you don't mind being really childish it can be managed. These are some of the games we played.

1. *Clumps*, you play in the usual way, only you think of a number and the questions are: " Is it odd or even ? is it a multiple of 3 ? has it so many digits ? is it a perfect square ?

2. *Snap*. According to the age of your players your snap cards are 15^2 and 225, 17^2 and 289, or 3×5 and 15, 2^6 and 64.

3. *Old Maid*, with algebra factors for forms just beginning algebra.

4. *Happy Family*. One family is conic sections, consisting of circle, parabola, ellipse and hyperbola; another—parallel lines, alternate $\angle$s, corresponding $\angle$s, and interior supplementary $\angle$s.

5. *Unit Making*. This is like word making and taking. You have lots of cardboard fractions $\frac{1}{2}$, $\frac{1}{4}$. $\frac{1}{8}$, $\frac{3}{8}$, $\frac{3}{4}$, etc., and when you see a whole number made up of these fractions, you shout and take it.

6. Tableaux.

Thales discovering that the angle in a semi-circle is a right angle.
Pythagoras making his mystic sign, the complete pentagon.
Archimedes and his circles.
Ptolemy demanding from Euclid a Royal Road to Geometry.

7. Nursery Rhymes, these are very easy to make and generally in a high moral tone—here is one :

Little Miss Ida
Worked out a rider
Writing plain and neat.
Along came an inspector
And seemed to expect her
To make it with reference complete.

and another :

Little Jack Horner sat in the corner
Using the ratio π
He wrote down the rule, they'd taught him at school,
A circle is $r^2\pi$.

Little Jack Horner came out of the corner
Shouting aloud Hurrah !
By my common sense, the circumference
I know is $2\pi r$.

When you come to set it down in words, it all sounds very little and trivial. But that some of the girls had been greatly interested is shown by the fact that I have had letters from old girls, who had evidently talked the Club over at College. They wrote asking if I could give them any information about books which might be helpful. Now I know of very few such books. Of course, Rouse Ball's *History of Mathematics*[1] and *Recreations*[2] are invaluable. Cajori's *History*[3] is very interesting. *The Story of Arithmetic*[4] by Susan Cunnington is useful, and White's *Scrap Book of Mathematics*[5] has lots of nice bits. *Rara Arithmetica*[6] is splendid.

I have still a few suggestions for further Club work.

1. A course on geometrical drawing as applied to Gothic arches, equilateral lancet, cusped and ogee, also geometrical foliage designs for stone work and tiles.

2. A collection of famous or striking mathematical sayings as :

(i) μηδεὶς ἀγεωμέτρητος εἰσίτω.
(ii) The truest things are things that never happened.
(iii) Give me a fulcrum and I will move the earth.
(iv) Here's to Pure Mathematics and may they never be of the least use to anybody.

3. The romance of π, beginning with Solomon, who according to 1 Kings vii. 23,[7] gave the value of π as 3 . . . down to Gregory, a Scotch professor, who in 1667, proved π to be incommensurable. In connection with this, I think girls are very interested if you show them that π, $\sqrt{2}$, $\sqrt{5}$ can be expressed geometrically, although it is impossible to do so arithmetically.

4. In *Rara Arithmetica*, you can find plenty to interest 14 year olds : there is Sacrobosco (what a beautiful name !) (1488)—his book was one of the first printed in Strasburg and it introduces, in Latin of course, that friend of everyone's—Thirty days hath September. . . . .

There are in Florentine books quaint illustrations of people finding heights of steeples and towers, drawings of tap screws, where the tank overflows badly.

In 1600 a book was printed in London after the style of Robert Recorde (1542-1558). The title page is amusing "The Arte of vulgar arithmeticke

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[1] 10s., Macmillan. [2] 8s. 6d., Macmillan. [3] 15s., Macmillan.
[4] 3s. 6d., Swan, Sonnenschein. [5] 5s. net, Open Court. [6] 4½\$, Ginn & Co.

[7] "And he made a molten sea, ten cubits from the one brim to the other: it was round all about . . . and a line of thirty cubits did compass it round about."

. . . the Rules, Precepts and Maxims are not only composed in meeter for the better retaining of them in memorie, but also the operations, examples, demonstrations and questions are in most easie wise expounded and explaned, in the form of a Dialogue, for the Readers more cleere understanding. A knowledge pleasant for Gentlemen, commendable for Capteines and Soldiers, profitable for merchants, and generally necessarie for all estates and degrees. Newly collected, digested, and in some part devised by a well-wisher to the Mathematicals." There follows a quotation from Ecclesiasticus, chap. 19: "Learning unto fooles is as fetters on their feete and manicles upon their right hand; but to the wise it is a jewell of golde, and like a Bracelet upon his right arme."

Whilst talking of ordinary sums in Arithmetic, I should like to read you two or three translations from the Greek.

Cypris thus addressed Love, who was looking downcast: "How, my child, hath sorrow fallen on thee"? And he answered: "The muses stole and divided among themselves in different proportions, the apples I was bringing from Helicon, snatching them from my bosom. Clio got the fifth part, Enterpe the twelfth, but divine Thalia the eighth. Melpomene carried off the twentieth part and Terpsichore the fourth and Erato the seventh; Polyhymnia robbed me of thirty apples and Urania of a hundred and twenty and Calliope went off with a load of three hundred apples. So I come to thee with lighter hands, bringing these fifty apples that the goddesses left me."

"We three Loves stand here pouring out water for the bath, sending streams into the fair flowing tank. I on the right, from my long-winged feet, fill it full in the sixth part of a day; I on the left, from my jar, fill it in four hours; and I in the middle, from my bow, in just half a day. Tell me in what a short time we should all fill it, pouring water from wings, bow and jar all at once."

5. There might be some reading and discussion on space. But the idea that Euclidean Geometry is not the one and only geometry rather staggers the schoolgirl; she is most indignant if after years of toiling at geometry, you whisper that the angles of $\triangle$ are *not* two right $\angle$s, and you cannot make much headway with either of the Two Non-Euclidean Geometries with their surfaces of positive and negative curvature. I generally find a talk on space is very humbling.

6. Graphs make a strong appeal to me—for the more advanced girls you can draw $y=\sin x$ and then on transparent paper the approximations

$$y=x, \quad y=x-\frac{x^3}{6}, \quad y=x-\frac{x^3}{6}+\frac{x^5}{120}\cdots$$

and you can almost see it breathe. Then can you ever in class get time to illustrate your curves and derived curves?

7. Many ideas are given in the *Mathematical Gazette*; in "Gleanings from Far and Near"; and if you unearth back copies there is "Mathematics and Morals." October, 1922, by C. M. Core, and "Greek Mathematics and Science," July, 1921, by Sir Thomas Heath.

Then there is Professor Garnett's Presidential Address of February, 1918, on "Alice Through the Convex Looking Glass."

By the bye there is a good deal of interest in the "Life and Letters of Lewis Carroll."[1]

I have kept to the last what would interest me most of all, if it could be done. If a musical mathematician, preferably a poet, would write a book on Rhythm, it would be well worth possessing. You could begin with Pythagoras, who as far as I know was the first to state that the various musical notes depend on number; that the Order and Beauty of the Universe have their origin in number; that the orderly motion of the Planets in their orbits creates the Music of the Spheres; that Love and Friendship are indicated by the Octave.

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[1] Fisher Unwin.

It was Pythagoras who, working with series of numbers, gave us Arithmetical, Geometrical, Harmonical Progressions, and the less known Musical Progression, whose terms are

$$a : \frac{2ab}{a+b} :: \frac{a+b}{2} : b,$$

$$e.g. \quad 6 : \ 8 \ :: \ 9 \ : 12.$$

Now a vibrating chord divided as

12 : 6, 12 : 8, 12 : 9 respectively

gives an octave, fifth and fourth.

Then Plato insisted that there was a liberal education in Rhythm and harmony, as given by Music and Mathematics.

Again, in Alfred Noyes, *The Torch Bearers*, Kepler is made to speak of "Those great rhythms that steer the moon and sun," and

"Have you not heard, in some great symphony,
Those golden mathematics making clear
The victory of the soul ?"

to Kepler also we owe

"The Laws of Nature are but the mathematical thoughts of God."

Then Sir Philip Sidney maintains that "the poet comes to you with words *set in delightful proportion*."

Finally, if by reason of the shortness of time at your disposal, you do none of these things, yet I would somehow and somewhere emphasize that the Spirit of Mathematics is Progress, and yet again Progress; that new discoveries have seldom disproved older theories; as fast as one simplification of methods, or discovery is made, another caps it. I would make clear that Geometry and its allied subjects progressed by leaps and bounds, while Arithmetic and calculations halted slowly, waiting for the Arabic Notation, the use of zero, the finding of logarithms. And each epoch-making discovery may be met with the remark: "I wonder nobody found it out before; when now known it is so easy."

I. M. BROWN.

SUMMER COURSES IN POST-GRADUATE MATHEMATICS.

THE second session of the Summer School for Post-Graduate Mathematics, organised by the Extra-mural Department of the University of Manchester, will be held at University College, Bangor, from Monday, 24th August, to Saturday, 5th September. The object of the school, which is recognised by the Board of Education, is to afford facilities for advanced study in Mathematics to teachers and others who have read Mathematics for a University degree.

The following three alternative courses are proposed, each one consisting of twenty lectures of one hour each, two lectures being taken on each of ten mornings:

(*a*) Atomic Structure and the Quantum Theory.
By Professor Sydney Chapman, M.A., D.Sc., F.R.S. (Imperial College of Science, London).

(*b*) Theory of Functions.
By Professor L. J. Mordell, F.R.S. (Manchester University).

(*c*) Higher Geometry.
By Mr. H. W. Richmond, M.A., F.R.S. (King's College, Cambridge).

Details of the fees, hostel accommodation and syllabuses may be had from Miss D. Withington, The University, Manchester. Applications should be made at an early date, as the holding of the courses depends to some extent upon the number of applications received.

MATHEMATICAL NOTES.

772. [C. 2. j.] *Direct Derivation of the Series for log 2.*

$$\log 2 = \int_1^2 \frac{1}{x}\,dx$$

$$= \lim_{n\to\infty} \frac{1}{n}\left[\frac{1}{1+\frac{1}{n}}+\frac{1}{1+\frac{2}{n}}+\dots+\frac{1}{1+\frac{n}{n}}\right]$$

$$= \lim_{n\to\infty}\left[\frac{1}{n+1}+\frac{1}{n+2}+\dots+\frac{1}{2n}\right].$$

But
$$\frac{1}{n+1}+\frac{1}{n+2}+\dots+\frac{1}{2n}$$

$$=1+\frac{1}{2}+\frac{1}{3}+\dots+\frac{1}{2n}-\left(\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\dots+\frac{1}{n}\right)$$

$$=1+\frac{1}{2}+\frac{1}{3}+\dots+\frac{1}{2n}-2\left(\frac{1}{2}+\frac{1}{4}+\frac{1}{6}+\dots+\frac{1}{2n}\right)$$

$$=1-\frac{1}{2}+\frac{1}{3}-\dots+\frac{1}{2n-1}-\frac{1}{2n};$$

$$\therefore\ \log 2 = \lim_{n\to\infty}\left(1-\frac{1}{2}+\frac{1}{3}+\dots+\frac{1}{2n-1}-\frac{1}{2n}\right)$$

$$=1-\frac{1}{2}+\frac{1}{3}-\dots$$

H. G. FORDER.

773. [A. 1. b.] *New proof of the theorem that the Arithmetic Mean of n positive numbers is greater than their Geometric Mean, if the numbers are not all equal.*

The theorem may be expressed in the form

$$\Sigma a^n > n\,.\,abc\dots k, \qquad\qquad (1)$$

where n is the number of the letters $a, b, c, \dots k$.

Assuming that (1) is true for some particular value of n, and introducing a new letter l, so as to make a total of $n+1$ letters, we have

$$l\overset{l}{\Sigma} a^n > n\,.\,abc\dots kl,$$

where $\overset{l}{\Sigma}$ denotes summation over all the letters *except l*.

Similarly
$$a\overset{a}{\Sigma} b^n > n\,.\,abc\dots kl,$$
$$b\overset{b}{\Sigma} a^n > \quad ,,$$
$$\dots\dots\dots\dots$$
$$k\overset{k}{\Sigma} a^n > \quad ,,$$

By addition, we get $\quad \Sigma a^n b > n(n+1)ab\dots kl. \qquad\qquad (2)$

But $\quad a^{n+1}+b^{n+1}-a^nb-b^na=(a^n-b^n)(a-b)>0$ unless $a=b$;

$$\therefore\ \Sigma(a^{n+1}+b^{n+1})>\Sigma(a^nb+b^na).$$

Now a^{n+1} occurs n times in $\Sigma(a^{n+1}+b^{n+1})$, and

$\Sigma(a^nb+b^na)$ is equivalent to Σa^nb;

$$\therefore\ n\Sigma a^{n+1} > \Sigma a^n b$$
$$> n(n+1)ab\dots kl, \text{ by (2)};$$
$$\therefore\ \Sigma a^{n+1} > (n+1)ab\dots kl.$$

Thus, if (1) is true for any particular value of n, it is true when that value is increased by 1.

Now when $n=2$, (1) becomes $a^2+b^2>2ab$, which is true if a and b are unequal. Hence, by induction (1) is true for all possible values of n.

If the numbers $a, b \ldots$ are all equal, the symbol $>$ must be replaced by $=$ throughout, and it is plain from the nature of the proof that if even one pair of the numbers are unequal, the inequation (1) holds good.

A similar method of proof can be applied to the theorem that the arithmetic mean is greater than the harmonic mean, and to other theorems on symmetrical functions of n letters.

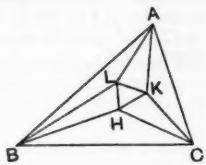
R. F. MUIRHEAD.

774. [K[1]. 1. c.] *On Note* 640, xi. p. 171.

$$\left.\begin{array}{lll} AK,\ AL & \text{trisect} & \angle A, \\ BL,\ BH & \quad ,, & \angle B, \\ CH,\ CK & \quad ,, & \angle C. \end{array}\right\} \text{ Show that } HKL \text{ is equilateral.}$$

$$\angle A\hat{L}B = 180^\circ - \frac{A+B}{3} = 120^\circ + \frac{C}{3}. \quad \text{(i)}$$

Also
$$\sin C = 3\sin\frac{C}{3} - 4\sin^3\frac{C}{3}$$
$$= 4\sin\frac{C}{3}\left(\sin^2 60^\circ - \sin^2\frac{C}{3}\right)$$
$$= 4\sin\frac{C}{3}\sin\left(60^\circ+\frac{C}{3}\right)\sin\left(60^\circ-\frac{C}{3}\right). \quad \text{(ii)}$$



Now
$$\frac{AL}{AB} = \frac{\sin\frac{B}{3}}{\sin ALB} = \frac{\sin\frac{B}{3}}{\sin\left(60^\circ-\frac{C}{3}\right)};$$

$$\therefore\ AL = \frac{AB\sin\frac{B}{3}}{\sin\left(60^\circ-\frac{C}{3}\right)} = \frac{2R\sin C\sin\frac{B}{3}}{\sin\left(60^\circ-\frac{C}{3}\right)}$$
$$= 8R\sin\frac{B}{3}\sin\frac{C}{3}\sin\left(60^\circ+\frac{C}{3}\right).$$

Similarly
$$AK = 8R\sin\frac{B}{3}\sin\frac{C}{3}\sin\left(60^\circ+\frac{B}{3}\right);$$

$$\therefore\ \frac{AL}{AK} = \frac{\sin\left(60^\circ+\frac{C}{3}\right)}{\sin\left(60^\circ+\frac{B}{3}\right)}.$$

Now
$$A\hat{K}L + A\hat{L}K = 180^\circ - \frac{A}{3} = \left(60^\circ+\frac{C}{3}\right)+\left(60^\circ+\frac{B}{3}\right).$$

Hence
$$A\hat{K}L = 60^\circ+\frac{C}{3} \quad \text{and} \quad A\hat{L}K = 60^\circ+\frac{B}{3}.*$$

Similarly
$$B\hat{L}H = 60^\circ+\frac{A}{3}.$$

But
$$A\hat{L}B = 120^\circ+\frac{C}{3};$$

$$\therefore\ H\hat{L}K = 60^\circ. \quad \text{Similarly for } L\hat{K}H \text{ and } K\hat{H}L.$$

E. M. LANGLEY.

* Since ratio of sines of two angles whose sum is a given magnitude $<180^\circ$ must be different for every partition into two angles.

775. [K¹. 21. b.] *Simple approximate construction for trisecting an acute angle ABC.*

Construct any rhombus $BCED$ having $A\hat{B}C$ for one of its angles. Produce DE, CE to F, L so that $EF = EL = \frac{1}{2}$ diagl. BE.

Since $$\sin FBC = 2 \sin EBF,$$

$$F\hat{B}C > 2E\hat{B}F,$$

and $$\therefore\ > \tfrac{2}{3}E\hat{B}C, \quad \textit{i.e.}\ > \tfrac{1}{3}A\hat{B}C.$$

The error is however very small, increasing from 0 to about 22′ as $A\hat{B}C$ increases from 0° to 90°.

For an angle of 78°, which is about the size of the $A\hat{B}C$,

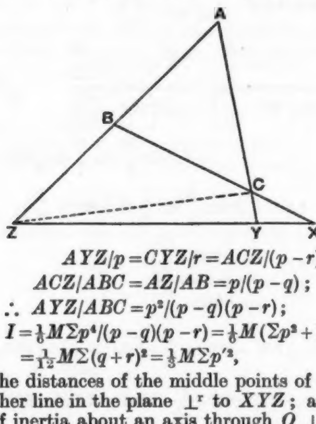
$$\begin{aligned}\cot FBC &= \tfrac{1}{2}\operatorname{cosec} 39° + \cot 39°\\ &= \cdot 7945 + 1\cdot 2349\\ &= 2\cdot 0294;\end{aligned}$$

$$\therefore\ F\hat{B}C = 26°\ 14' \text{ to nearest minute.}$$

The construction is due to Mr. J. J. Gorham. E. M LANGLEY.

776. [R. 2. c.] *Moment of Inertia of a Triangle about any Line in its Plane.*

The moment of inertia of a triangle about a side is $Mh^2/6$, where h is the altitude. In the figure triangle ABC is equal to the algebraic sum of the triangles AYZ, BZX, CXY. Hence the moment of inertia (I) about XYD is $(\Sigma AYZ\,.\,p^2)\sigma/6$, where σ is the density and p, q, r the distances of A, B, C from XYZ.



Now $$AYZ/p = CYZ/r = ACZ/(p-r)$$

and $$ACZ/ABC = AZ/AB = p/(p-q);$$

$$\therefore\ AYZ/ABC = p^2/(p-q)(p-r);$$

$$\begin{aligned}\therefore\ I &= \tfrac{1}{6}M\Sigma p^4/(p-q)(p-r) = \tfrac{1}{6}M(\Sigma p^2 + \Sigma pr)\\ &= \tfrac{1}{12}M\Sigma(q+r)^2 = \tfrac{1}{3}M\Sigma p'^2,\end{aligned}$$

where p', q', r' are the distances of the middle points of the sides from XYZ.

Similarly for another line in the plane $\perp^r$ to XYZ; and if these lines meet in O, the moment of inertia about an axis through O $\perp^r$ to the plane of the $\triangle$ is $M(\Sigma OA'^2)/3$, where A', B', C' are the middle points of the sides; and in particular $$I_G = M(\Sigma GA'^2/3) = M\Sigma a^2/36.$$

Again, if the figure be turned upside down and ABC be a triangle immersed vertically in a liquid and intersecting the free surface in XYZ, the depth of the centre of pressure is

$$\begin{aligned}&\tfrac{1}{2}(\Sigma AYZ\,.\,p^2)/\Sigma AYZ\,.\,p\\ &= \Sigma\{(p^4/(p-q)(p-r)\}/\Sigma\{p^3/(p-q)(p-r)\}\\ &= \tfrac{1}{2}(\Sigma p^2 + \Sigma qr)/\Sigma p.\end{aligned}$$

N. M. GIBBINS.

REVIEWS.

Mathematics for Technical Students. By E. B. VERITY. Pp. xi-468. 12s. 6d. 1924. (Longmans, Green & Co).

This book follows the now well-recognised lines of books on Practical Mathematics. It contains fourteen chapters devoted to Algebra, seven devoted to Numerical Trigonometry, including the graphs of periodic functions and short sections on the cycloid and epicycloid, and eleven chapters on the elementary processes of the Calculus. The Calculus portion consists of applications of the differential coefficient and of the integral of x^n. The differential of x^n when n is a positive integer is made to depend on the binomial theorem, and no proof of the result for other values of n is given, although the results are extensively used. It seems a pity that one of the now well-known elementary proofs of the differential coefficient of x^n for all values of the index should not have been included, and in a book of this size one might reasonably expect applications of the differential coefficients of the trigonometrical functions. There is a natural tendency to avoid general proofs; for example, $\frac{dy}{dt}=\frac{dy}{dx}\cdot\frac{dx}{dt}$ is proved only for the case of $y=x^2$, and is then used generally; but, in fairness to the author, it must be stated that it is made perfectly clear whenever a result is given without proof. The book is very clearly and carefully written; it is well printed and arranged, and can confidently be recommended to those students who, at the beginning of their Practical Mathematics course, are prepared to buy a book covering the work for three years.

A. D.

Felix Klein. Elementarmathematik vom höheren Standpunkte aus. 1 Bd. *Arithmetik, Algebra, Analysis.* Ausgearbeitet von E. HELLINGER. 3 Auflage mit Zusätzen von F. SEYFARTH. Pp. viii + 321. Goldmark 15. 1924. (Springer, Berlin.)

The reform movement, for which Klein and his friends stand, is in principle that of our Association. The difficulties of teaching mathematics are the same all the world over, half inherent in the subject and half in human nature. There is the universal demand for more periods for mathematics in the time-table, and if the examination bogey does not exist in Germany, his place is more than filled by the Government curriculum, which has at times absolutely forbidden the teaching of the calculus, and driven the poor teacher to all sorts of deceit.

The German demands for an advanced point of view in school analysis go, perhaps, beyond our own, but Klein also stands for the fullest possible use of *Anschaulichkeit*, which is not quite the same as intuition. This includes an interplay of geometry and analysis; on the other hand, they are only at the very beginning of any correlation with physics: "applied mathematics" means commercial arithmetic and drawing for engineers.

Klein's main assault is on the great gulf fixed between what a man learns at the university and teaches in school, and on the "hysteresis" between progress in mathematical theory and in school practice: an interval of some decades is reasonable, but not of some centuries. His method is to take a series of isolated chapters, definitely beyond school range, from the construction of a calculating machine to the continuity of the continuum. Each is expounded, as only Klein can, with very little assumption of previous knowledge, and is a joy and a delight in itself, apart from the didactic application. He goes on to show why the teacher should know about it, and how it ought to affect his lessons on the elementary parts of the same subject.

The work stands much as it did in the first edition of 1908; two Appendices bring up to date the accounts of the history and literature of the subject. It does not need explaining why the intervening period has been barren.

Mathematicians of all branches, and especially teachers, will find it a fascinating and a stirring book.

H. P. H.

Space and Time. By C. Benedicks. Translated by J. Harden. Pp. xiv + 98. 4s. 1924. (Methuen.)

Prof. Benedicks is not lacking in courage! He proposes to abandon the wave theory of light and to return to a modified emission theory. To understand his object it is necessary to recall the starting point of the theory of relativity. This theory may be considered to have arisen in the attempt to reconcile three apparently contradictory results:

(i) the phenomenon of astronomical aberration, which seemed to show that the ether did *not* share in the motion of the earth in its orbit;

(i) Fizeau's water-tube experiment, which seemed to show that the ether was *partially*, but not wholly, carried along by a moving body;

(iii) the Michelson-Morley experiment, which seemed to show that the ether was *wholly* carried along by the earth in its motion.

These results showed that the ideas concerning the ether conceived by Faraday, Maxwell, and Kelvin, were in need of revision. These ideas had been very useful in the past, as they had suggested results which experiment had proved to be true, but a stage was reached when some of the suggested results turned out to be false. These erroneous predictions were not confined to optics. It was predicted that if an electric charge were at rest in the ether, a moving instrument would experience no magnetic effect. Experiment showed that there *was* an effect, which depended only on the *relative* motion of the charge and instrument.

Einstein's theory does not deny the existence of the ether, but it certainly implies that for the discussion of many phenomena it can be ignored as a superfluous hypothesis. The ether, formerly dominant in physics, is now left out of account when physical effects are to be calculated. It is like a monarch reduced from absolute power to the position of a mere crowned figure-head. However, although the idea of an ether is relegated to the background, the wave theory of light, or rather its analytical expression by means of Maxwell's equations, is fully retained, or even increased in importance. This may seem illogical to some physicists, but the mathematician will easily recognise that we may take Maxwell's equations as our starting point, without making any assumptions about a medium, and deduce all the results of physical optics. This is Einstein's method, in his 1905 paper *On the Electrodynamics of Moving Bodies.*

Prof. Benedicks considers that this procedure arises from "an exaggerated respect for the Maxwell-Lorentz theory of light in its present form, combined with a somewhat scant respect for the fundamental conceptions in themselves," and that Einstein's theories show "a contempt, which is carried extremely far, for our hitherto accepted modes of reasoning." It is claimed that "the traditional fundamental conceptions of space and time are quite clear and vigorous." To support this claim synchronism of distant clocks is defined by means of an imaginary "absolutely solid body," which transmits movements instantaneously. The fact that no such bodies exist seems a distinct defect in this definition. When one end of a real solid is moved, elastic waves travel along it, and it takes a finite time, although of course a very short one, before the motion is communicated to the other end.

It is recognised that if Einstein's theory is rejected, the optical experiments that gave rise to that theory call for an alternative explanation. Prof. Benedicks puts forward a modified emission theory as adequate to meet all the requirements. "All the optical phenomena concerned receive a full qualitative explanation through a possibility first pointed out by Ritz; that is, the abandonment of the obscure conception of the ether, which gives rise to all the difficulties, and the substitution of light regarded as an emission in space." Of course an emission theory gives the simplest possible explanation of aberration and of the Michelson-Morley result. It is not so easy to deal with the Fizeau water-tube experiment on these lines, but Menges (*Phil. Mag.*, March 1925) believes that he has accomplished this satisfactorily.

But any emission theory has several serious difficulties to overcome. Newton's emission theory was abandoned, because to explain the facts of refraction it had to assume that the velocity of light increased with the density

of the medium, an assumpion which Foucault showed to be untrue. Prof. Benedicks attempts to evade this obstacle by assuming that Huygens' principle holds good in the emission theory. As it is generally considered that this principle is closely connected with the wave theory, this assumption will need a great deal of justification before it can be accepted. Another difficulty is that an emission theory requires the velocity of light to be dependent upon that of its source. De Sitter considers that observations on double stars prove that there is no such dependence, and Majorana believes that he has given a direct experimental proof of de Sitter's conclusions. Prof. Benedicks quotes Freundlich's criticism of de Sitter's arguments, but ignores Majorana's experiments, perhaps because of the doubt entertained by many physicists as to whether these experiments prove what they are supposed to prove. In general, Prof. Benedicks relies upon the work of Ritz, and does not go into details himself. A sketch is given of an attempt to explain polarisation, but there is no mention of diffraction phenomena, which are usually claimed as striking examples of the accuracy of the wave theory. In fact, the book is not much more than a collection of roughly outlined suggestions. It is possible that these suggestions may ultimately be developed, but until this has been done, and until an emission theory has shown itself able to account for the great number of experimental facts that fit in so well with the wave theory, the proposal to abandon the latter for the former is not likely to be seriously considered. The book is to be commended because it is interesting and likely to stimulate thought on the problems it discusses, even though its conclusions may not immediately commend themselves. A cautious preface by Sir Oliver Lodge points out that in the present state of flux of physical science, " it is not safe to turn down any contribution of a thoughtful and reasonable physicist who is acquainted with the facts."

Einstein's Theory of Relativity. By M. Born. Translated by H. L. Brose. Pp. xi + 293. 12s. 1924. (Methuen.)

Relativity for Physics Students. By G. B. Jeffery. Pp. vii + 151. 6s. 1924. (Methuen.)

The Theory of Relativity. By A. Henderson, A. W. Hobbs, and J. W. Lasley, Jr. Pp. xiii + 99. 11s. 6d. 1924. (University of North Carolina Press, per Oxford University Press.)

Much of the difficulty in understanding the theory of relativity arises from a lack of familiarity with the physical ideas and experimental results that preceded and suggested that theory. Prof. Born has made a brilliant attempt to include in a single volume all the preliminary knowledge of mechanics and physics required to understand Einstein's work, as well as an account of the work itself. The reader is not required to know any mechanics or physics, and the only mathematics used is a little algebra and geometry of a very elementary character. It is amazing with what skill the author sketches out the main features of dynamics, optics, and electrodynamics, emphasising all the time the aspects that lead naturally to the idea of relativity.

The book is an elaboration of lectures given in Germany to a large audience, apparently composed of the general public. It is to be feared that in England the " man in the street " is not as patient and docile as his Teutonic counterpart. One cannot repress grave doubts as to whether the English general reader will work through the long preliminary matter which occupies about two-thirds of the book. The exclusive use of very elementary methods has the effect of making some arguments unduly long. For example, the footnote which takes up about half of pp. 226-227 might be compressed to a quarter of its length by the use of calculus. Those who are weak in mathematics will probably shirk any proof that is at all long, however elementary, and they will almost certainly be baffled by a few very unfortunate misprints.

If some portions are too long, others are too short. The perihelion of Mercury receives only about a page of rather vague discussion, which gives very little idea of how the results were obtained.

However, apart from these points, the book has some excellent features. Prof. Born gives a very interesting explanation of a supposed paradox, which

some have used as an argument against the validity of the theory of relativity. "Let us consider an observer A at rest at the origin O of the inertial system S. A second observer B is at first to be at rest at the same point O, and is then to move off with uniform velocity along a straight line, say the x-axis, until he has reached a point C, when he is to turn round and return to O along a straight line with the same velocity . . . then the clock of the observer B must have lost time compared with the clock of A. . . . According to the theory of relativity two systems in relative motion are equivalent. We may therefore also regard B as at rest. A then performs a journey in exactly the same way as B previously, but in the opposite direction. We must therefore conclude that when A returns B's clock is in advance of A's. But previously we had come to exactly the opposite conclusion." Prof. Born points out that the fallacy of this argument lies in applying the *restricted* theory, which is valid only for non-accelerated systems, to the motion of B, which *is* accelerated. When the general theory is applied, as it should be, B's clock is found to be in advance of A's, and the apparent contradiction vanishes.

Among other features that call for commendation, we may mention the very clear explanation of the measurement of the curvature of a surface by surveying operations, such as can be actually carried out without anything but measurements on the surface itself. There are 135 diagrams, and an excellent portrait of Einstein.

Prof. Jeffery's book, as its title implies, is not for "the man in the street" or for the expert mathematician, but for an intermediate class, for "students of science who are able to make some use of mathematics as an instrument of thought, but who may be not quite ready to face the mathematical analysis which is essential for the thorough exploration of the subject in all its ramifications." However, the introduction, which occupies about one-fifth of the book, should be quite intelligible to those who have hardly any knowledge of mathematics at all. It traces the evolution of the ideas of mechanics, starting with the sixteenth-century astronomer Tycho Brahe. His observations were studied by his assistant Kepler, who succeeded in deducing from them three fairly simple laws of planetary motion. Meanwhile Galileo had ascertained by experiment the laws which govern the motion of falling bodies. The work of these men provided the material that enabled Newton to set up his laws of motion, which, except for a few minute discrepancies, have proved adequate to explain and predict the motion of the solar system. But Newton's "absolute space" raised a great difficulty. This space may be roughly defined as that mapped out by the fixed stars. Unfortunately for this definition the so-called fixed stars are, in some cases at least, moving relatively to each other with widely different velocities. A new hope of defining a frame of reference that should have some claim to the title absolute arose with the progress of the ether theory, which in the hands of Faraday and Maxwell proved so valuable in electricity and optics. But this hope was soon blighted. "If mechanics adopted the ether in order to simplify the problem of motion, never was foster-parent blessed with a more unruly child . . . the time had arrived when some fundamental reconstruction of the theory could no longer be delayed. Einstein did not bring forth his theory merely as an elaboration and refinement of physical law in order to bring theory into accord with a few isolated and newly discovered facts: he brought it forth to meet the situation created by a complete theoretical breakdown of the older system."

The book then proceeds to discuss the restricted principle of relativity, which is treated fairly fully, including the application to aberration, the Doppler effect, Fresnel's dragging coefficient, and the field of a moving electron. At this stage the reader must be prepared to face the equations of the electromagnetic field as adopted by Lorentz. Of course, every serious student of physics should be familiar with these, but it is to be feared that many who are attracted chiefly by the experimental side of the subject will be rather frightened by the number of partial differential coefficients involved.

The general theory is dealt with very briefly. Prof. Jeffery's treatment of the gravitational deflection of light and of the spectral shift is very simple, and seems valuable for beginners, although, as he frankly admits, it is open to criticism at several points. The motion of the perihelion of Mercury gives

more trouble. The usual formula for the interval is quoted without proof, but with a brief outline of the principles used to obtain it. The orbit is then deduced by making the interval between any two points stationary. The work is given in full, and can be understood by those who are entirely ignorant of the calculus of variations.

Prof. Jeffrey concludes with a list of books and papers for those who desire a more thorough treatment. "Every young English physicist should study at least the first of these books" (viz. Eddington's *Report to the Physical Society on the Relativity Theory of Gravitation*). There is good ground for the hope expressed "that these lectures will help to smooth the way by a preliminary exploration of the ground."

Three members of the mathematical staff of the University of North Carolina have compiled a little book, which is rather in the nature of a set of lecture notes. The account of the general theory follows Eddington's work very closely; in fact, some pages are almost verbatim quotations from his *Mathematical Theory of Relativity*. The fourth and last chapter of the book under review deals with the curvature of manifolds, and appears to have been summarised from well-known works on Differential Geometry. This bringing together in a compact form of the most important results of much larger books is a useful piece of work, though the price is rather high for about a hundred pages. But what is to be regretted is that the book is described in a sub-title as "Studies and Contributions," while the preface speaks of "special investigations by the respective authors." These phrases seem to imply original research, and are out of place in a compilation of this character.

However, if the book is judged for what it is, rather than what it professes to be, it makes quite a favourable impression. On pp. 67-68 an account is given of the Lick Observatory eclipse expedition of Sept. 1922, and the statement is made (as has been widely published) that the average deflection obtained agreed exactly with Einstein's prediction. But a footnote says that an examination of a greater number of plates has given a larger value (2″·05 instead of 1″·725). This statement does not seem to have appeared elsewhere (at any rate in England), and it would be interesting to know if it is confirmed.

H. T. H. Piaggio.

Hydrodynamics. By H. Lamb. 5th edn. Pp. xvi+687. 45s. net. 1924. (Cambridge University Press.)

At the present time there is probably no other English book on Applied Mathematics which occupies a position like Lamb's *Hydrodynamics*. It is a text-book and a book of reference, a standard treatise used throughout the whole world, and if the value of a book be estimated by the number of times it is quoted, then the value of this book is above measure. It is the book to which one always goes first for an indication of the present state of knowledge on any branch of the subject, and its five editions have kept it thoroughly up to date. The author's great power of seizing on the essential features of a new piece of work and of putting it into its due place as a part of the whole subject has been, and is, of great assistance both to those who desire to apply results and to those who wish to extend them.

Though embodying the research of all nationalities, it exemplifies the best traditions of British work, and forms a lucid unification of the great contributions of Stokes, Kelvin and Rayleigh.

The history of the book reflects the rich history of the subject for the last fifty years, and it is therefore of interest to note the nature of the additions which have been made on the occasion of each new edition.

The first appeared in 1879 as a comparatively small book, under the title *Treatise on the Motion of Fluids*. Its general character, as compared with later editions, was more purely mathematical. The specialised part of the book consisted of chapters on waves in liquids, waves in air and viscosity, and occupied about a quarter of the whole.

With the second edition of 1895 the book assumed its present name, and took on the leading features which it has since maintained. Types of hydrodynamical problems were brought into relation with general dynamics, and with Lagrange's method in particular. Much was added on the motion of

solids through an infinite liquid, and a separate chapter was devoted to tidal waves. This chapter, which involved much important original work, has ever since formed what is probably the best account of the elements of the dynamical theory of the tides. To the chapter on surface-waves much was added on the problem of Cauchy and Poisson, while new work, such as the theory of group-velocity, was, of course, included. Waves of finite amplitude were considered in various connections, and a chapter was added on rotating masses of liquid in space.

The third edition of 1906 contained, among many other new things, accounts of the work of the author himself on non-harmonic two-dimensional wave motion, of Hough on the theory of the tides, of Rayleigh on the diffraction of sound-waves by small obstacles, of Osborne Reynolds on turbulence, and of Poincaré on rotating masses of liquid.

The fourth edition appeared in 1916, and among the editions we may perhaps specially mention a section on the wave-making resistance of ships, a simple version of Sommerfeld's diffraction problem, the author's own work on atmospheric motions, and that of Oseen on the motion of solids through viscous liquids. An indication of the bridging of the important gap between the subject of theoretical hydrodynamics and certain parts of the empirical sublect of hydraulics was given by a section on the resistance of fluids to the relative motion of solids.

In the fifth edition, which is now before us, what we have called the specialised part of the book occupies two-thirds of the whole. Of the additions to the early chapters two deal with the theory of the Pitot-tube, one with Kutta and Jaukowski's example of two-dimensional irrotational motion with circulation, designed for application to the theory of aerofoils, and another to the adiabatic flow of a gas.

Sections are added on the radial motion of an infinite liquid with a spherical cavity and on the motion of a cylinder through a liquid possessing vorticity. It well illustrates Professor Lamb's scientific and practical point of view to note that though problems on spherical cavities have figured for a long time in examination papers, he has waited until he could quote original work by Rayleigh in 1917 on the collapse of a spherical bubble, and by himself in 1923 on the early stages of a submarine explosion.

The chapter on tidal waves contains an original theorem, with several applications, extending to rotating systems, Rayleigh's approximate method of calculating periods of free oscillation. The chapter on waves of expansion has been amplified by Rankine's investigation on waves of permanent type.

The important additions are nost numerous, however, in the chapter on viscosity, where they include the following:

1. Viscous motion between nearly parallel planes in connection with the theory of lubrication.
2. The diffusion of vorticity, with an original bit of work on currents due to a suddenly started surface traction, and a sketch of Ekman's theory of the action of wind on ocean currents.
3. Rayleigh's theory on the damping of sound in narrow channels.
4. Motion between rotating cylinders, with an indication of the remarkable results obtained by G. I. Taylor in 1921.
5. Eddy-viscosity, and a sketch of Taylor's theoretical work on the variation of wind with altitude over the Earth's surface.
6. A sketch of recent work on the resistance of fluids.

To the chapter on rotating masses of liquid, general theorems of Poincaré (1885) and Lichtenstein (1918) have been added.

The present edition is a little more closely printed than the earlier ones, but the general appearance of the book remains excellent. J. P.

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315. Sir Isaac Newton used to say that the longitude would be discovered either by a fool or an accident.—Francis Moore, Physician, *The Age of Intellect*, p. 124, footnote (1819).

316. Euclid would run a bad chance if a Russell contradicted him.—Sydney Smith to Lord John Russell.

THE LIBRARY.

160 Castle Hill, Reading.

ADDITIONS.

The Librarian reports the following gifts:

From Mr. **W. J. Greenstreet**:

J. Wallis — A Treatise of Algebra, both Historical and Practical, shewing, The Original, Progress, and Advancement thereof, from time to time; and by what Steps it hath attained to the Heighth at which now it is. With some Additional Treatises, I. Of the Cono-Cuneus.... II. Of Angular Sections.... III. Of the Angle of Contact.... IV. Of Combinations.... - - - - - - 1685

The 'additional treatises' have their own title pages, in three cases with the date 1684, and their own paging, but they are included in the general table of contents. Caswell's *Doctrine of Trigonometry, both Plain and Spherical* was 'adjoined' by Wallis to the volume.

J. Ward — Lives of the Gresham Professors - - - - - 1740

From Sir **Thomas Muir**:

Messenger of Mathematics, Vols. 28-30 (1898-1901).

The number for March 1901 (*No.* 359) *was missing, and* Dr. Glaisher *has very kindly replaced it.*

From Prof. **E. H. Neville**:

I. Barrow — Euclide's *Elements* and *Data* {8} - - - - - 1722

P. Ghosh — Euclid's Elements of Geometry; I {3: A. S. Ghosh} 1895

An Indian school book, printed in Calcutta.

From Miss **B. A. L. Oswald**:

School books by F. R. Barrell, H. S. Hall and F. H. Stevens, and A. F. van der Heyden.

From Mr. **C. Tweedie**:

A. L. Andreini — Sfere Cosmografiche - - - (Hoepli 376-377) 1907

E. Pascal — Calcolo delle Variazioni e Calcolo delle Differenze Finite - - - - - - (Hoepli 248-249) 1897

Funzioni Ellittiche - - - (Hoepli 210) 1896

Gruppi Continui di Trasformazioni (Hoepli 327-328) 1903

S. Pincherle — Geometria Pura {4} - - - - (Hoepli 32) 1895

F. Virgilii e C. Garibaldi — Economia Matematica - - - (Hoepli 281) 1899

Also nine papers by N. Nielsen.

From Mr. **R. M. Wright**:

C. V. Durell — Plane Geometry; I - - - - - - - 1909

Part II was in the Godfrey collection.

E. M. Radford — Problem Papers - - - - - - - - 1904

A REQUEST.

A member wishes to borrow Somerville's *Rhythmic Approach to Mathematics*.

Donations of back numbers of the "Gazette" are always welcome.

YORKSHIRE BRANCH.

A WELL-ATTENDED meeting of the Yorkshire Branch of the Mathematical Association was held on Saturday, 7th February, at the Refectory of the University of Leeds.

Professor Brodetsky was in the chair.

During private business it was announced that the Committee proposed that the Society should celebrate the Bi-Centenary of the death of Sir Isaac Newton, who died on 20th March, 1727. A paper was read by Mr. G. Stanley Bishop, of the West Leeds High School, on "Wooden Methods for Wooden Heads," in which he explained some methods adopted in teaching arithmetic to backward pupils in South Africa. This was followed by an interesting address by Mr. C. W. Gilham, Lecturer in Mathematics at the University of Leeds, on "Mechanics before Newton." The lecture was illustrated by lantern slides, and traced the development of mechanical principles from the time of Aristotle till the end of the seventeenth century.

REPORT OF THE SYDNEY BRANCH OF THE MATHEMATICAL ASSOCIATION FOR THE YEAR 1924.

THERE are now 24 members of the parent Association, and 83 associate members.

The distribution of the *Gazette* has proceeded satisfactorily.

There have been three meetings held during the year. At the first meeting Mr. Meldrum spoke on "Some Impressions of the Position of Mathematics in English Schools." At the second meeting the Association was fortunate in having an address from Prof. D. M. Y. Sommerville, of Victoria University College, on "The Circle at Infinity." This was much appreciated.

At the annual meeting Miss Janet Brown, B.Sc., gave an interesting address on her experiences as a teacher in one of the London schools.

The Office-Bearers for 1925 were elected as follows: *President*, Prof. H. S. Carslaw; *Hon. Treasurer*, Mr. A. B. Colville, B.A.; *Joint Hon. Secretaries*, Miss Janet Brown, B.Sc.; Mr. H. J. Meldrum, B.A., B.Sc.

ERRATA.

P. 329, l. 15. *For* Biographical *read* Bibliographical.

P. 341, l. 11. *For* Hereford *read* Hertford.

P. 348, l. 19. *For* or $\left(\frac{16}{41}-\frac{1}{r^2}\right)\ldots$ *read* or $-\left(\frac{16}{41}-\frac{1}{r^2}\right)\ldots$.

l. 20. *For* $\frac{128}{r^4}$ *read* $\frac{128}{41^2}$.

P. 349, last line. *For* Lowry *read* Lucy.

PRINTED IN GREAT BRITAIN BY ROBERT MACLEHOSE AND CO LTD.
AT THE UNIVERSITY PRESS, GLASGOW

THE MATHEMATICAL ASSOCIATION.

(*An Association of Teachers and Students of Elementary Mathematics.*)

"*I hold every man a debtor to his profession, from the which as men of course do seek to receive countenance and profit, so ought they of duty to endeavour themselves by way of amends to be a help and an ornament thereunto.*"—BACON.

THE MATHEMATICAL ASSOCIATION, which was founded in 1871, as the *Association for the Improvement of Geometrical Teaching*, aims not only at the promotion of its original object, but at bringing within its purview all branches of elementary mathematics.

Its purpose is to form a strong combination of all persons who are interested in promoting good methods of teaching mathematics. The Association has already been largely successful in this direction. It has become a recognised authority in its own department, and has exerted an important influence on methods of examination.

The Annual Meeting of the Association is held in January. Other Meetings are held when desired. At these Meetings papers on elementary mathematics are read and discussed.

Branches of the Association have been formed in London, Bangor, Yorkshire, Bristol, Manchester, Cardiff, Sydney (New South Wales), and Queensland (Brisbane). Further information concerning these branches can be obtained from the Honorary Secretaries of the Association.

"**The Mathematical Gazette**" (published by Messrs. G. BELL & SONS, LTD.) is the organ of the Association. It is issued at least six times a year. The price per copy (to non-members) is usually 2s. 6d. each. The *Gazette* contains—

(1) ARTICLES, mainly on subjects within the scope of elementary mathematics;

(2) NOTES, generally with reference to shorter and more elegant methods than those in current text-books;

(3) REVIEWS, written by men of eminence in the subject of which they treat. They deal with the more important English and Foreign publications, and their aim, where possible, is to dwell on the general development of the subject, as well as upon the part play ed therein by the book under notice;

(4) QUERIES AND ANSWERS, on mathematical topics of a general character.

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